

NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AT RESONANCE WITH NONLINEAR WENTZELL BOUNDARY CONDITIONS

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Dedicated to the 70th birthday of Jerome A. Goldstein

ABSTRACT. In the first part of the article, we give necessary and sufficient conditions for the solvability of a class of nonlinear elliptic boundary value problems with nonlinear boundary conditions involving the q -Laplace-Beltrami operator. In the second part, we give some additional results on existence and uniqueness and we study the regularity of the weak solutions for these classes of nonlinear problems. More precisely, we show some global a priori estimates for these weak solutions in an L^∞ -setting.

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^N$, $N \geq 1$, be a bounded domain with a Lipschitz boundary $\partial\Omega$ and consider the following nonlinear boundary value problem with nonlinear second order boundary conditions:

$$\begin{cases} -\Delta_p u + \alpha_1(u) = f(x), & \text{in } \Omega, \\ b(x)|\nabla u|^{p-2} \partial_{\mathbf{n}} u - \rho b(x) \Delta_{q,\Gamma} u + \alpha_2(u) = g(x), & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $b \in L^\infty(\partial\Omega)$, $b(x) \geq b_0 > 0$, for some constant b_0 , ρ is either 0 or 1, and $\alpha_1, \alpha_2 \in C(\mathbf{R}, \mathbf{R})$ are monotone nondecreasing functions such that $\alpha_i(0) = 0$. Moreover, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplace operator, $p \in (1, +\infty)$ and $f \in L^2(\Omega, dx)$, $g \in L^2(\partial\Omega, \sigma)$ are given real-valued functions. Here, dx denotes the usual N -dimensional Lebesgue measure in Ω and σ denotes the restriction to $\partial\Omega$ of the $(N-1)$ -dimensional Hausdorff measure. Recall that σ coincides with the usual Lebesgue surface measure since Ω has a Lipschitz boundary, and $\partial_{\mathbf{n}} u$ denotes the normal derivative of u in direction of the outer normal vector $\vec{\mathbf{n}}$. Furthermore, $\Delta_{q,\Gamma}$ is defined as the generalized q -Laplace-Beltrami operator on $\partial\Omega$, that is, $\Delta_{q,\Gamma} u = \operatorname{div}_\Gamma(|\nabla_\Gamma u|^{q-2} \nabla_\Gamma u)$, $q \in (1, +\infty)$. In particular, $\Delta_2 = \Delta$ and $\Delta_{2,\Gamma} = \Delta_\Gamma$ become the well-known Laplace and Laplace-Beltrami operators on Ω and $\partial\Omega$, respectively. Here, for any real valued function v ,

$$\operatorname{div}_\Gamma v = \sum_{i=1}^{N-1} \partial_{\tau_i} v,$$

where $\partial_{\tau_i} v$ denotes the directional derivative of v along the tangential directions τ_i at each point on the boundary, whereas $\nabla_\Gamma v = (\partial_{\tau_1} v, \dots, \partial_{\tau_{N-1}} v)$ denotes the tangential gradient at $\partial\Omega$. It is worth mentioning again that when $\rho = 0$ in (1.1), the boundary conditions are

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of lower order than the order of the p -Laplace operator, while for $p = 1$, we deal with boundary conditions which have the same differential order as the operator acting in the domain Ω . Such boundary conditions arise in many applications, such as phase-transition phenomena (see, e.g., [13, 14] and the references therein) and have been studied by several authors (see, e.g., [2, 12, 16, 24, 28]).

In a recent paper [12], the authors have formulated necessary and sufficient conditions for the solvability of (1.1) when $p = q = 2$, by establishing a sort of "nonlinear Fredholm alternative" for such elliptic boundary value problems. We shall now state their main result. Defining two real parameters $\lambda_1, \lambda_2 \in \mathbb{R}_+$ by

$$\lambda_1 = \int_{\Omega} dx, \quad \lambda_2 = \int_{\partial\Omega} \frac{d\sigma}{b}, \quad (1.2)$$

this result reads that a necessary condition for the existence of a weak solution of (1.1) is that

$$\int_{\Omega} f(x) dx + \int_{\partial\Omega} g(x) \frac{d\sigma}{b(x)} \in (\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)), \quad (1.3)$$

while a sufficient condition is

$$\int_{\Omega} f(x) dx + \int_{\partial\Omega} g(x) \frac{d\sigma}{b(x)} \in \text{int}(\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)), \quad (1.4)$$

where $\mathcal{R}(\alpha_j)$ denotes the range of α_j , $j = 1, 2$ and $\text{int}(G)$ denotes the interior of the set G .

Relation (1.3) turns out to be both necessary and sufficient if either of the sets $\mathcal{R}(\alpha_1)$ or $\mathcal{R}(\alpha_2)$ is an open interval. This particular result was established in [12, Theorem 3], by employing methods from convex analysis involving subdifferentials of convex, lower semicontinuous functionals on suitable Hilbert spaces. As an application of our results, we can consider the following boundary value problem

$$\begin{cases} -\Delta u + \alpha_1(u) = f(x), & \text{in } \Omega, \\ b(x) \partial_n u = g(x), & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

which is only a special case of (1.1) (i.e., $p = 0$, $\alpha_2 \equiv 0$ and $p = 2$). According to [12, Theorem 3] (see also (1.4)), this problem has a weak solution if

$$\int_{\Omega} f(x) dx + \int_{\partial\Omega} g(x) \frac{d\sigma}{b(x)} \in \text{int}(\lambda_1 \mathcal{R}(\alpha_1)), \quad (1.6)$$

which yields the result of Landesman and Lazer [17] for $g \equiv 0$. This last condition is both necessary and sufficient when the interval $\mathcal{R}(\alpha_1)$ is open. This was put into an abstract context and significantly extended by Brezis and Haraux [8]. Their work was much further extended by Brezis and Nirenberg [9]. The goal of the present article is comparable to that of [12] since we want to establish similar conditions to (1.4) and (1.6) for the existence of solutions to (1.1) when $p, q \neq 2$, with main emphasis on the generality of the boundary conditions.

Recall that λ_1 and λ_2 are given by (1.2). Let \mathbb{I} be the interval $\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)$. Our first main result is as follows (see Section 4 also).

Theorem 1.1. *Let $\alpha_j : \mathbb{R} \rightarrow \mathbb{R}$ ($j = 1, 2$) be odd, monotone nondecreasing, continuous function such that $\alpha_j(0) = 0$. Assume that the functions $\Lambda_j(t) := \int_0^{|t|} \alpha_j(s) ds$ satisfy*

$$\Lambda_j(2t) \leq C_j \Lambda_j(t), \quad \text{for all } t \in \mathbb{R}, \quad (1.7)$$

for some constants $C_j > 1$, $j = 1, 2$. If u is a weak solution of (1.1) (in the sense of Definition 4.10 below), then

$$\int_{\Omega} f(x) dx + \int_{\partial\Omega} g(x) \frac{d\sigma}{b(x)} \in \mathbb{I}. \quad (1.8)$$

Conversely, if

$$\int_{\Omega} f(x) dx + \int_{\partial\Omega} g(x) \frac{d\sigma}{b(x)} \in \text{int}(\mathbb{I}), \quad (1.9)$$

then (1.1) has a weak solution.

Our second main result of the paper deals with a modified version of (1.1) which is obtained by replacing the functions $\alpha_1(s)$, $\alpha_2(s)$ in (1.1) by $\bar{\alpha}_1(s) + |s|^{p-2}s$ and $\bar{\alpha}_2(s) + \rho b|u|^{q-2}u$, respectively, and also allowing $\bar{\alpha}_1$, $\bar{\alpha}_2$ to depend on $x \in \bar{\Omega}$. Under additional assumptions on $\bar{\alpha}_1$, $\bar{\alpha}_2$ and under higher integrability properties for the data (f, g) , the next theorem provides us with conditions for unique solvability results for solutions to such boundary value problems. Then, we obtain some regularity results for these solutions. In addition to these results, the continuous dependence of the solution to (1.1) with respect to the data (f, g) can be also established. In particular, we prove the following

Theorem 1.2. *Let all the assumptions of Theorem 1.1 be satisfied for the functions $\bar{\alpha}_1$, $\bar{\alpha}_2$. Moreover, for each $j = 1, 2$, assume that $\bar{\alpha}_j(t)/t \rightarrow 0$, as $t \rightarrow 0$ and $\bar{\alpha}_j(t)/t \rightarrow \infty$, as $t \rightarrow \infty$, respectively.*

(a) *Then, for every $(f, g) \in L^{p_1}(\Omega) \times L^{q_1}(\partial\Omega)$ with*

$$p_1 > \max \left\{ 1, \frac{N}{p} \right\}, \quad q_1 > \begin{cases} \max \left\{ 1, \frac{N-1}{p-1} \right\}, & \text{if } p \in \{0, 1\}, \\ \max \left\{ 1, \frac{N-1}{p} \right\}, & \text{if } p = 1 \text{ and } p = q, \end{cases}$$

there exists a unique weak solution to problem (1.1) (in the sense of Definition 5.3 below) which is bounded.

(b) *Let $\bar{\alpha}_j$, $j = 1, 2$, be such that*

$$c_j |\bar{\alpha}_j(\xi - \eta)| \leq |\bar{\alpha}_j(\xi) - \bar{\alpha}_j(\eta)|, \quad \text{for all } \xi, \eta \in \mathbb{R},$$

for some constants $c_j \in (0, 1]$. Then, the weak (bounded) solution of problem (1.1) depends continuously on the data (f, g) . Precisely, let us indicate by u_{F_j} the unique solution corresponding to the data $F_j := (f_j, g_j) \in L^{p_1}(\Omega) \times L^{q_1}(\partial\Omega)$, for each $j = 1, 2$. Then, the following estimate holds:

$$\|u_{F_1} - u_{F_2}\|_{L^\infty(\Omega)} + \|u_{F_1} - u_{F_2}\|_{L^\infty(\partial\Omega)} \leq Q(\|f_1 - f_2\|_{L^{p_1}(\Omega)}, \|g_1 - g_2\|_{L^{q_1}(\partial\Omega)}),$$

for some nonnegative function $Q : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $Q(0, 0) = 0$, which can be computed explicitly.

We organize the paper as follows. In Section 2, we introduce some notations and recall some well-known results about Sobolev spaces, maximal monotone operators and Orlicz type spaces which will be needed throughout the article. In Section 3, we show that the subdifferential of a suitable functional associated with problem (1.1) satisfies a sort of "quasilinear" version of the Fredholm alternative (cf. Theorem 3.5), which is needed in order to obtain the result in Theorem 1.1. Finally, in Sections 4 and 5, we provide detailed proofs of Theorem 1.1 and Theorem 1.2. We also illustrate the application of these results with some examples.

2. PRELIMINARIES AND NOTATIONS

In this section we put together some well-known results on nonlinear forms, maximal monotone operators and Sobolev spaces. For more details on maximal monotone operators, we refer to the monographs [4, 7, 20, 21, 27]. We will also introduce some notations.

2.1. Maximal monotone operators. Let H be a real Hilbert space with scalar product $(\cdot, \cdot)_H$.

Definition 2.1. Let $A : D(A) \subset H \rightarrow H$ be a closed (nonlinear) operator. The operator A is said to be:

(i) monotone, if for all $u, v \in D(A)$ one has

$$(Au - Av, u - v)_H \geq 0.$$

(ii) maximal monotone, if it is monotone and the operator $I + A$ is invertible.

Next, let V be a real reflexive Banach space which is densely and continuously embedded into the real Hilbert space H , and let V' be its dual space such that $V \hookrightarrow H \hookrightarrow V'$.

Definition 2.2. Let $\mathcal{A} : V \times V \rightarrow \mathbb{R}$ be a continuous map.

(a) The map $\mathcal{A} : V \times V \rightarrow \mathbb{R}$ is called a nonlinear form on H if for all $u \in V$ one has $\mathcal{A}(u, \cdot) \in V'$, that is, if \mathcal{A} is linear and bounded in the second variable.

(b) The nonlinear form $\mathcal{A} : V \times V \rightarrow \mathbb{R}$ is said to be:

(i) monotone if $\mathcal{A}(u, u - v) - \mathcal{A}(v, u - v) \geq 0$ for all $u, v \in V$;

(ii) hemicontinuous if $\lim_{t \downarrow 0} \mathcal{A}(u + tv, w) = \mathcal{A}(u, w)$, $\forall u, v, w \in V$;

(iii) coercive, if $\lim_{\|v\|_V \rightarrow +\infty} \frac{\mathcal{A}(v, v)}{\|v\|_V} = +\infty$.

Now, let $\varphi : H \rightarrow (-\infty, +\infty]$ be a proper, convex, lower semicontinuous functional with effective domain

$$D(\varphi) := \{u \in H : \varphi(u) < \infty\}.$$

The subdifferential $\partial\varphi$ of the functional φ is defined by

$$\begin{cases} D(\partial\varphi) &:= \{u \in D(\varphi) : \exists w \in H \forall v \in D(\varphi) : \varphi(v) - \varphi(u) \geq (w, v - u)_H\}; \\ \partial\varphi(u) &:= \{w \in H : \forall v \in D(\varphi) : \varphi(v) - \varphi(u) \geq (w, v - u)_H\}. \end{cases}$$

By a classical result of Minty [20] (see also [7, 21]), $\partial\varphi$ is a maximal monotone operator.

2.2. Functional setup. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. For $1 < p < \infty$, we let $W^{1,p}(\Omega)$ be the first order Sobolev space, that is,

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N\}.$$

Then $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\|u\|_{\Omega,p}^p + \|\nabla u\|_{\Omega,p}^p \right)^{1/p}$$

is a Banach space, where we have set

$$\|u\|_{\Omega,p}^p := \int_{\Omega} |u|^p dx.$$

Since Ω has a Lipschitz boundary, it is well-known that there exists a constant $C > 0$ such that

$$\|u\|_{\Omega,p_s} \leq C \|u\|_{W^{1,p}(\Omega)}, \text{ for all } u \in W^{1,p}(\Omega), \quad (2.1)$$

where $p_s = \frac{pN}{N-p}$ if $p < N$, and $1 \leq p_s < \infty$ if $N = p$. Moreover the trace operator $\text{Tr}(u) := u|_{\partial\Omega}$ initially defined for $u \in C^1(\bar{\Omega})$ has an extension to a bounded linear operator from $W^{1,p}(\Omega)$ into $L^{q_s}(\partial\Omega)$ where $q_s := \frac{p(N-1)}{N-p}$ if $p < N$, and $1 \leq q_s < \infty$ if $N = p$. Hence, there is a constant $C > 0$ such that

$$\|u\|_{\partial\Omega, q_s} \leq C\|u\|_{W^{1,p}(\Omega)}, \text{ for all } u \in W^{1,p}(\Omega). \quad (2.2)$$

Throughout the remainder of this article, for $1 < p < N$, we let

$$p_s := \frac{pN}{N-p} \text{ and } q_s := \frac{p(N-1)}{N-p}. \quad (2.3)$$

If $p > N$, one has that

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{N}{p}}(\bar{\Omega}), \quad (2.4)$$

that is, the space $W^{1,p}(\Omega)$ is continuously embedded into $C^{0,1-\frac{N}{p}}(\bar{\Omega})$. For more details, we refer to [23, Theorem 4.7] (see also [19, Chapter 4]).

For $1 < q < \infty$, we define the Sobolev space $W^{1,q}(\partial\Omega)$ to be the completion of the space $C^1(\partial\Omega)$ with respect to the norm

$$\|u\|_{W^{1,q}(\partial\Omega)} := \left(\int_{\partial\Omega} |u|^q d\sigma + \int_{\partial\Omega} |\nabla_{\Gamma} u|^q d\sigma \right)^{1/q},$$

where we recall that $\nabla_{\Gamma} u$ denotes the tangential gradient of the function u at the boundary $\partial\Omega$. It is also well-known that $W^{1,q}(\partial\Omega)$ is continuously embedded into $L^{q_t}(\partial\Omega)$ where $q_t := \frac{q(N-1)}{N-1-q}$ if $1 < q < N-1$, and $1 \leq q_t < \infty$ if $q = N-1$. Hence, for $1 < q \leq N-1$, there exists a constant $C > 0$ such that

$$\|u\|_{q_t, \partial\Omega} \leq C\|u\|_{W^{1,q}(\partial\Omega)}, \text{ for all } u \in W^{1,q}(\partial\Omega). \quad (2.5)$$

Let λ_N denote the N -dimensional Lebesgue measure and let the measure $\mu := \lambda_N|_{\Omega} \oplus \sigma$ on $\bar{\Omega}$ be defined for every measurable set $A \subset \bar{\Omega}$ by

$$\mu(A) := \lambda_N(\Omega \cap A) + \sigma(A \cap \partial\Omega).$$

For $p, q \in [1, \infty]$, we define the Banach space

$$X^{p,q}(\bar{\Omega}, \mu) := \{F = (f, g) : f \in L^p(\Omega) \text{ and } g \in L^q(\partial\Omega)\}$$

endowed with the norm

$$\|F\|_{X^{p,q}(\bar{\Omega})} = \|F\|_{p,q} := \|f\|_{\Omega, p} + \|g\|_{\partial\Omega, q},$$

if $1 \leq p, q < \infty$, and

$$\|F\|_{X^{\infty, \infty}(\bar{\Omega}, \mu)} = \|F\|_{\infty} := \max\{\|f\|_{\Omega, \infty}, \|g\|_{\partial\Omega, \infty}\}.$$

If $p = q$, we will simply denote $\|F\|_{p,p} = \|F\|_p$.

Identifying each function $u \in W^{1,p}(\Omega)$ with $U = (u, u|_{\partial\Omega})$, we have that $W^{1,p}(\Omega)$ is a subspace of $X^{p,p}(\bar{\Omega}, \mu)$.

For $1 < p, q < \infty$, we endow

$$\mathcal{V}_1 := \{U := (u, u|_{\partial\Omega}), u \in W^{1,p}(\Omega), u|_{\partial\Omega} \in W^{1,q}(\partial\Omega)\}$$

with the norm

$$\|U\|_{\mathcal{V}_1} := \|u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,q}(\partial\Omega)},$$

while

$$\mathcal{V}_0 := \{U = (u, u|_{\partial\Omega}) : u \in W^{1,p}(\Omega)\}$$

is endowed with the norm

$$\|U\|_{\mathcal{V}_0} := \|u\|_{W^{1,p}(\Omega)}.$$

It follows from (2.1)-(2.2) that \mathcal{V}_0 is continuously embedded into $X^{p_s, q_s}(\overline{\Omega}, \mu)$, with p_s and q_s given by (2.3), for $1 < p < N$. Moreover, by (2.1) and (2.5), \mathcal{V}_1 is continuously embedded into $X^{p_s, q_t}(\overline{\Omega}, \mu)$.

2.3. Musielak-Orlicz type spaces. For the convenience of the reader, we introduce the Orlicz and Musielak-Orlicz type spaces and prove some properties of these spaces which will be frequently used in the sequel (see Section 5).

Definition 2.3. Let (X, Σ, ν) be a complete measure space. We call a function $B : X \times \mathbb{R} \rightarrow [0, \infty]$ a Musielak-Orlicz function on X if

- (a) $B(x, \cdot)$ is non-trivial, even, convex for ν -a.e. $x \in X$;
- (b) $B(x, \cdot)$ is vanishing and continuous at 0 for ν -a.e. $x \in X$;
- (c) $B(x, \cdot)$ is left continuous on $[0, \infty)$;
- (d) $B(\cdot, t)$ is Σ -measurable for all $t \in [0, \infty)$;
- (e) $\lim_{t \rightarrow \infty} \frac{B(x, t)}{t} = \infty$.

The complementary Musielak-Orlicz function \tilde{B} is defined by

$$\tilde{B}(x, t) := \sup\{s|t| - B(x, s) : s > 0\}.$$

It follows directly from the definition that for $t, s \geq 0$ (and hence for all $t, s \in \mathbb{R}$)

$$st \leq B(x, t) + \tilde{B}(x, s).$$

Definition 2.4. We say that a Musielak-Orlicz function B satisfies the (Δ_α^0) -condition ($\alpha > 1$) if there exists a set X_0 of ν -measure zero and a constant $C_\alpha > 1$ such that

$$B(x, \alpha t) \leq C_\alpha B(x, t),$$

for all $t \in \mathbb{R}$ and every $x \in X \setminus X_0$.

We say that B satisfies the (∇_2^0) -condition if there is a set X_0 of ν -measure zero and a constant $c > 1$ such that

$$B(x, t) \leq \frac{1}{2c} B(x, ct),$$

for all $t \in \mathbb{R}$ and all $x \in X \setminus X_0$.

Definition 2.5. A function $\Phi : \mathbb{R} \rightarrow [0, \infty)$ is called an \mathcal{N} -function if

- Φ is even, strictly increasing and convex;
- $\Phi(t) = 0$ if and only if $t = 0$;
- $\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0$ and $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \infty$.

We say that an \mathcal{N} -function Φ satisfies the (Δ_2) -condition if there exists a constant $C_2 > 1$ such that

$$\Phi(2t) \leq C_2 \Phi(t), \quad \text{for all } t \in \mathbb{R},$$

and it satisfies the (∇_2) -condition if there is a constant $c > 1$ such that

$$\Phi(t) \leq \Phi(ct)/(2c), \quad \text{for all } t \in \mathbb{R}.$$

For more details on \mathcal{N} -functions, we refer to the monograph of Adams [1, Chapter VIII] (see also [25, Chapter I], [26, Chapter I]).

Remark 2.6. For an \mathcal{N} -function Φ , we let φ be its left-sided derivative. Then φ is left continuous on $(0, \infty)$ and nondecreasing. Let ψ be given by

$$\psi(s) := \inf\{t > 0 : \varphi(t) > s\}.$$

Then

$$\Phi(t) = \int_0^{|t|} \varphi(s) ds; \quad \Psi(t) := \int_0^{|t|} \psi(s) ds = \sup\{|t|s - \Phi(s) : s > 0\}.$$

As before for all $s, t \in \mathbb{R}$,

$$st \leq \Phi(t) + \Psi(s). \quad (2.6)$$

Moreover, if $s = \varphi(t)$ or $t = \psi(s)$ then we have equality, that is,

$$\Psi(\varphi(t)) = t\varphi(t) - \Phi(t). \quad (2.7)$$

The function Ψ is called the complementary \mathcal{N} -function of Φ . It is also known that an \mathcal{N} -function Φ satisfies the (Δ_2) -condition if and only if

$$ct\varphi(t) \leq \Phi(t) \leq t\varphi(t), \quad (2.8)$$

for some constant $c \in (0, 1]$ and for all $t \in \mathbb{R}$, where φ is the left-sided derivative of Φ .

Lemma 2.7. *Let Φ be an \mathcal{N} -function which satisfies the (Δ_2) -condition with the constant $C_2 > 1$ and let Ψ be its complementary \mathcal{N} -function. Then Ψ satisfies the (∇_2) -condition with the constant $c := 2^{C_2-1}$.*

Proof. We have

$$t\varphi(t) \leq \int_t^{2t} \varphi(s) ds \leq \int_0^{2t} \varphi(s) ds = \Phi(2t) \leq C_2\Phi(t).$$

Since $\varphi(\psi(s)) \geq s$ for all $s \geq 0$ and $s/\Psi(s)$ and $s/(s-1)$ are decreasing, we get for $t := \psi(s)$, that

$$\frac{s\psi(s)}{\Psi(s)} \geq \frac{\varphi(\psi(s))\psi(s)}{\Psi(\varphi(\psi(s)))} = \frac{t\varphi(t)}{\Psi(\varphi(t))} = \frac{t\varphi(t)}{t\varphi(t) - \Phi(t)} \geq \frac{C_2}{C_2 - 1}.$$

Now let $c := 2^{C_2-1}$. Then for $t \geq 0$,

$$\begin{aligned} \ln\left(\frac{\Psi(ct)}{\Psi(t)}\right) &= \int_t^{ct} \frac{\psi(s)}{\Psi(s)} ds \geq \int_t^{ct} \frac{C_2}{s(C_2 - 1)} ds \\ &= \frac{C_2}{C_2 - 1} \ln(c) = C_2 \log(2) = \ln(2 \cdot 2^{C_2-1}). \end{aligned}$$

Hence, $\Psi(t)2c \leq \Psi(ct)$. □

Corollary 2.8. *Let B be a Musielak-Orlicz function such that $B(x, \cdot)$ is an \mathcal{N} -function for v -a.e. x . If B satisfies the (Δ_2^0) -condition, then \tilde{B} satisfies the (∇_2^0) -condition.*

Definition 2.9. *Let B be a Musielak-Orlicz function. Then the Musielak-Orlicz space $L^B(X)$ associated with B is defined by*

$$L^B(X) := \{u : X \rightarrow \mathbb{R} \text{ measurable} : \rho_B(u/\alpha) < \infty \text{ for some } \alpha > 0\},$$

where

$$\rho_B(v) := \int_X B(x, v(x)) dv(x).$$

On this space we consider the Luxemburg norm $\|\cdot\|_{X,B}$ defined by

$$\|u\|_{X,B} := \inf\{\alpha > 0 : \rho_B(u/\alpha) \leq 1\}.$$

Proposition 2.10. *Let B be a Musielak-Orlicz function which satisfies the (∇_2^0) -condition. Then*

$$\lim_{\|u\|_{X,B} \rightarrow +\infty} \frac{\rho_B(u)}{\|u\|_{X,B}} = +\infty.$$

Proof. If B satisfies the (∇_2^0) -condition, then there exists a set $X_0 \subset X$ of measure zero such that for every $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) > 0$,

$$B(x, \alpha t) \leq \alpha \varepsilon B(x, t), \quad (2.9)$$

for all $t \in \mathbb{R}$ and all $x \in X \setminus X_0$. Let $\lambda \in (0, \infty)$ be fixed. For $\varepsilon := 1/\lambda$ there exists $\alpha > 0$ satisfying the above inequality. We will show that $\rho_B(u) \geq \lambda \|u\|_{X,B}$ whenever $\|u\|_{X,B} > 1/\alpha$. Assume that $\|u\|_{X,B} > 1/\alpha$ and let $\delta > 0$ be such that $\alpha = (1 + \delta)/\|u\|_{X,B}$. Then

$$\begin{aligned} \rho_B(\alpha u) &= \int_X B(x, u(1 + \delta)/\|u\|_{X,B}) d\mu \\ &\geq (1 + \delta)^{1-1/n} \int_X B(x, u(1 + \delta)^{1/n}/\|u\|_{X,B}) d\mu \geq (1 + \delta)^{1-1/n}, \end{aligned}$$

for all $n \in \mathbb{N}$. If we assume that the last inequality does not hold, then

$$\|u\|_{X,B}/(1 + \delta) \in \{\alpha > 0 : \rho(u/\alpha) \leq 1\},$$

and this clearly contradicts the definition of $\|u\|_{X,B}$. Therefore, we must have

$$\rho_B(\alpha u) \geq 1 + \delta = \alpha \|u\|_{X,B}. \quad (2.10)$$

From (2.9), (2.10), we obtain

$$\rho_B(u) = \int_X B(x, u(x)) d\mu \geq \frac{\lambda}{\alpha} \int_X B(x, \alpha u(x)) d\mu = \frac{\lambda}{\alpha} \rho_B(\alpha u) \geq \lambda \|u\|_{X,B}.$$

The proof is finished. \square

Corollary 2.11. *Let B be a Musielak-Orlicz function such that $B(x, \cdot)$ is an \mathcal{N} -function for v -a.e. x . If its complementary \mathcal{N} -function \tilde{B} satisfies the (Δ_2^0) -condition, then B satisfies the (∇_2^0) -condition and*

$$\lim_{\|u\|_{X,B} \rightarrow +\infty} \frac{\rho_B(u)}{\|u\|_{X,B}} = +\infty.$$

2.4. Some tools. For the reader's convenience, we report here below some useful inequalities which will be needed in the course of investigation.

Lemma 2.12. *Let $a, b \in \mathbb{R}^N$ and $p \in (1, \infty)$. Then, there exists a constant $C_p > 0$ such that*

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq C_p (|a| + |b|)^{p-2} |a - b|^2 \geq 0. \quad (2.11)$$

If $p \in [2, \infty)$, then there exists a constant $c_p \in (0, 1]$ such that

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq c_p |a - b|^p. \quad (2.12)$$

Proof. The proof of (2.12) is included in [10, Lemma I.4.4]. In order to show (2.11), one only needs to show that the left hand side is non-negative, which follows easily. \square

The following result which is of analytic nature and whose proof can be found in [22, Lemma 3.11] will be useful in deriving some a priori estimates of weak solutions of elliptic equations.

Lemma 2.13. *Let $\psi : [k_0, \infty) \rightarrow \mathbb{R}$ be a non-negative, non-increasing function such that there are positive constants c, α and δ ($\delta > 1$) such that*

$$\psi(h) \leq c(h-k)^{-\alpha} \psi(k)^\delta, \quad \forall h > k \geq k_0.$$

Then $\psi(k_0 + d) = 0$ with $d = c^{1/\alpha} \psi(k_0)^{(\delta-1)/\alpha} 2^{\delta(\delta-1)}$.

3. THE FREDHOLM ALTERNATIVE

In what follows, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$. Let $b \in L^\infty(\partial\Omega)$ satisfy $b(x) \geq b_0 > 0$ for some constant b_0 . Let \mathbb{X}_2 be the real Hilbert space $L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{d\sigma}{b})$. Then, it is clear that \mathbb{X}_2 is isomorphic to $X^{2,2}(\overline{\Omega}, \lambda_N \oplus \sigma)$ with equivalent norms.

Next, let $\rho \in \{0, 1\}$ and $p, q \in (1, +\infty)$ be fixed. We define the functional $\mathcal{J}_\rho : \mathbb{X}_2 \rightarrow [0, +\infty]$ by setting

$$\mathcal{J}_\rho(U) = \begin{cases} \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{q} \int_{\partial\Omega} \rho |\nabla_\Gamma u|^q d\sigma, & \text{if } U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_\rho), \\ +\infty, & \text{if } U \in \mathbb{X}_2 \setminus D(\mathcal{J}_\rho), \end{cases} \quad (3.1)$$

where the effective domain is given $D(\mathcal{J}_\rho) = \mathcal{V}_\rho \cap \mathbb{X}_2$.

Throughout the remainder of this section, we let $\mu := \lambda_N \oplus \frac{d\sigma}{b}$. The following result can be obtained easily.

Proposition 3.1. *The functional \mathcal{J}_ρ defined by (3.1) is proper, convex and lower semicontinuous on $\mathbb{X}_2 = X^{2,2}(\overline{\Omega}, \mu)$.*

The following result contains a computation of the subdifferential $\partial \mathcal{J}_\rho$ for the functional \mathcal{J}_ρ .

Remark 3.2. Let $U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_\rho)$ and let $F := (f, g) \in \partial \mathcal{J}_\rho(U)$. Then, by definition, $F \in \mathbb{X}_2$ and for all $V = (v, v|_{\partial\Omega}) \in D(\mathcal{J}_\rho)$, we have

$$\int_{\overline{\Omega}} F(V - U) d\mu \leq \frac{1}{p} \int_\Omega (|\nabla v|^p - |\nabla u|^p) dx + \frac{1}{q} \int_{\partial\Omega} \rho (|\nabla_\Gamma v|^q - |\nabla_\Gamma u|^q) d\sigma.$$

Let $W = (w, w|_{\partial\Omega}) \in D(\mathcal{J}_\rho)$, $0 < t \leq 1$ and set $V := tW + U$ above. Dividing by t and taking the limit as $t \downarrow 0$, we obtain that

$$\int_{\overline{\Omega}} FW d\mu \leq \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w dx + \rho \int_{\partial\Omega} |\nabla_\Gamma u|^{q-2} \nabla_\Gamma u \cdot \nabla_\Gamma w d\sigma, \quad (3.2)$$

where we recall that

$$\int_{\overline{\Omega}} F d\mu = \int_\Omega f dx + \int_{\partial\Omega} g \frac{d\sigma}{b}.$$

Choosing $w = \pm \psi$ with $\psi \in \mathcal{D}(\Omega)$ (the space of test functions) and integrating by parts in (3.2), we obtain

$$-\Delta_p u = f \quad \text{in } \mathcal{D}'(\Omega)$$

and

$$g = b(x) |\nabla u|^{p-2} \partial_n u - \rho b(x) \Delta_{q,\Gamma} u \quad \text{weakly on } \partial\Omega.$$

Therefore, the single valued operator $\partial \mathcal{J}_\rho$ is given by

$$D(\partial \mathcal{J}_\rho) = \{U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_\rho), \left(-\Delta_p u, b(x) |\nabla u|^{p-2} \partial_n u - \rho b(x) \Delta_{q,\Gamma} u \right) \in \mathbb{X}_2\},$$

and

$$\partial \mathcal{J}_\rho(U) = \left(-\Delta_p u, b(x) |\nabla u|^{p-2} \partial_n u - \rho b(x) \Delta_{q,\Gamma} u \right). \quad (3.3)$$

□

Since the functional \mathcal{J}_ρ is proper, convex and lower semicontinuous, it follows that its subdifferential $\partial \mathcal{J}_\rho$ is a maximal monotone operator.

In the following two lemmas, we establish a relation between the null space of the operator $A_\rho := \partial \mathcal{J}_\rho$ and its range.

Lemma 3.3. *Let $\mathcal{N}(A_\rho)$ denote the null space of the operator A_ρ . Then*

$$\mathcal{N}(A_\rho) = C\mathbf{1} = \{C = (c, c) : c \in \mathbb{R}\},$$

that is, $\mathcal{N}(A_\rho)$ consists of all the real constant functions on $\overline{\Omega}$.

Proof. We say that $U \in \mathcal{N}(A_\rho)$ if and only if (by definition) $U = (u, u|_{\partial\Omega})$ is a weak solution of

$$\begin{cases} -\Delta_p u = 0, & \text{in } \Omega, \\ b(x) |\nabla u|^{p-2} \partial_n u - \rho b(x) \Delta_{q,\Gamma} u = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

A function $U = (u, u|_{\partial\Omega}) \in \mathcal{V}_\rho \cap \mathbb{X}_2$ is said to be a weak solution of (3.4), if for every $V = (v, v|_{\partial\Omega}) \in \mathcal{V}_\rho \cap \mathbb{X}_2$, there holds

$$\mathcal{A}_\rho(U, V) := \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \rho \int_{\partial\Omega} |\nabla_\Gamma u|^{q-2} \nabla_\Gamma u \cdot \nabla_\Gamma v \, d\sigma = 0. \quad (3.5)$$

Let $C := (c, c)$ with $c \in \mathbb{R}$. Then it is clear that $C \in \mathcal{N}(A_\rho)$.

Conversely, let $U = (u, u|_{\partial\Omega}) \in \mathcal{N}(A_\rho)$. Then, it follows from (3.5) that

$$\mathcal{A}_\rho(U, U) := \int_{\Omega} |\nabla u|^p \, dx + \rho \int_{\partial\Omega} |\nabla_\Gamma u|^q \, d\sigma = 0. \quad (3.6)$$

Since Ω is bounded and connected, this implies that u is equal to a constant. Therefore, $U = C\mathbf{1}$ and this completes the proof. □

Lemma 3.4. *The range of the operator A_ρ is given by*

$$\mathcal{R}(A_\rho) = \left\{ F := (f, g) \in \mathbb{X}_2 : \int_{\Omega} F \, d\mu := \int_{\Omega} f \, dx + \int_{\partial\Omega} g \frac{d\sigma}{b(x)} = 0 \right\}.$$

Proof. Let $F \in \mathcal{R}(A_\rho) \subset \mathbb{X}_2$. Then there exists $U = (u, u|_{\partial\Omega}) \in D(A_\rho)$ such that $A_\rho(U) = F$. More precisely, for every $V = (v, v|_{\partial\Omega}) \in \mathcal{V}_\rho \cap \mathbb{X}_2$, we have

$$\begin{aligned} \mathcal{A}_\rho(U, V) &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \rho \int_{\partial\Omega} |\nabla_\Gamma u|^{q-2} \nabla_\Gamma u \cdot \nabla_\Gamma v \, d\sigma \\ &= \int_{\Omega} FV \, d\mu. \end{aligned} \quad (3.7)$$

Taking $V = (1, 1) \in \mathcal{V}_\rho \cap \mathbb{X}_2$, we obtain that $\int_{\Omega} F \, d\mu = 0$. Hence,

$$\mathcal{R}(A_\rho) \subseteq \left\{ F \in \mathbb{X}_2 : \int_{\Omega} F \, d\mu = 0 \right\}.$$

Let us now prove the converse. To this end, let $F \in \mathbb{X}_2$ be such that $\int_{\Omega} F d\mu = 0$. We have to show that $F \in \mathcal{R}(A_\rho)$, that is, there exists $U \in \mathcal{V}_\rho \cap \mathbb{X}_2$ such that (3.7) holds, for every $V \in \mathcal{V}_\rho \cap \mathbb{X}_2$. To this end, consider

$$\mathcal{V}_{\rho,0} := \left\{ U = (u, u|_{\partial\Omega}) \in \mathcal{V}_\rho \cap \mathbb{X}_2 : \int_{\Omega} U d\mu := \int_{\Omega} u dx + \int_{\partial\Omega} u \frac{d\sigma}{b} = 0 \right\}.$$

It is clear that $\mathcal{V}_{\rho,0}$ is a closed linear subspace of $\mathcal{V}_\rho \cap \mathbb{X}_2 \hookrightarrow \mathbb{X}_2$, and therefore is a reflexive Banach space. Using [18, Section 1.1], we have that the norm

$$\|U\|_{\mathcal{V}_{\rho,0}} := \|\nabla u\|_{p,\Omega} + \rho \|\nabla_\Gamma u\|_{q,\partial\Omega}$$

defines an equivalent norm on $\mathcal{V}_{\rho,0}$. Hence, there exists a constant $C > 0$ such that for every $U \in \mathcal{V}_{\rho,0}$,

$$\|U\|_2 \leq C \|U\|_{\mathcal{V}_{\rho,0}} := \|\nabla u\|_{p,\Omega} + \rho \|\nabla_\Gamma u\|_{q,\partial\Omega}. \quad (3.8)$$

Define the functional $\mathcal{F}_\rho : \mathcal{V}_{\rho,0} \rightarrow \mathbb{R}$ by

$$\mathcal{F}_\rho(U) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{\rho}{q} \int_{\partial\Omega} |\nabla_\Gamma u|^q d\sigma - \int_{\Omega} F U d\mu.$$

It is easy to see that \mathcal{F}_ρ is convex and lower-semicontinuous on \mathbb{X}_2 (see Proposition 3.1). We show now that \mathcal{F}_ρ is coercive. By exploiting a classical Hölder inequality and using (3.8), we have

$$\begin{aligned} \left| \int_{\Omega} F U d\mu \right| &\leq C \|F\|_2 \|U\|_2 \leq C \|F\|_2 \|U\|_{\mathcal{V}_{\rho,0}} \\ &= C \|F\|_2 (\|\nabla u\|_{p,\Omega} + \rho \|\nabla_\Gamma u\|_{q,\partial\Omega}). \end{aligned}$$

Obviously, this estimate yields

$$- \int_{\Omega} F U d\mu \geq -C \|F\|_2 (\|\nabla u\|_{p,\Omega} + \rho \|\nabla_\Gamma u\|_{q,\partial\Omega}). \quad (3.9)$$

Therefore, from (3.9), we immediately get

$$\frac{\mathcal{F}_\rho(U)}{\|U\|_{\mathcal{V}_{\rho,0}}} \geq \frac{\frac{1}{p} \|\nabla u\|_{p,\Omega}^p + \frac{\rho}{q} \|\nabla_\Gamma u\|_{q,\partial\Omega}^q}{\|\nabla u\|_{p,\Omega} + \rho \|\nabla_\Gamma u\|_{q,\partial\Omega}} - C \|F\|_2.$$

This inequality implies that

$$\lim_{\|U\|_{\mathcal{V}_{\rho,0}} \rightarrow +\infty} \frac{\mathcal{F}_\rho(U)}{\|U\|_{\mathcal{V}_{\rho,0}}} = +\infty,$$

and this shows that the functional \mathcal{F}_ρ is coercive. Since \mathcal{F}_ρ is also convex, lower-semicontinuous, it follows from [3, Theorem 3.3.4] that, there exists a function $U^* \in \mathcal{V}_{\rho,0}$ which minimizes \mathcal{F}_ρ . More precisely, for all $V \in \mathcal{V}_{\rho,0}$, $\mathcal{F}_\rho(U^*) \leq \mathcal{F}_\rho(V)$; this implies that for every $0 < t \leq 1$ and every $V \in \mathcal{V}_{\rho,0}$,

$$\mathcal{F}_\rho(U^* + tV) - \mathcal{F}_\rho(U^*) \geq 0.$$

Hence,

$$\lim_{t \downarrow 0} \frac{\mathcal{F}_\rho(U^* + tV) - \mathcal{F}_\rho(U^*)}{t} \geq 0.$$

Using the Lebesgue Dominated Convergence, an easy computation shows that

$$0 \leq \lim_{t \downarrow 0} \frac{\mathcal{F}_\rho(U^* + tV) - \mathcal{F}_\rho(U^*)}{t} = \int_{\Omega} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx + \rho \int_{\partial\Omega} |\nabla_\Gamma u^*|^{q-2} \nabla_\Gamma u^* \cdot \nabla_\Gamma v \, d\sigma - \int_{\Omega} FV \, d\mu. \quad (3.10)$$

Changing V to $-V$ into (3.10) gives that

$$\int_{\Omega} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx + \rho \int_{\partial\Omega} |\nabla_\Gamma u^*|^{q-2} \nabla_\Gamma u^* \cdot \nabla_\Gamma v \, d\sigma = \int_{\Omega} FV \, d\mu, \quad (3.11)$$

for every $V \in \mathcal{V}_{\rho,0}$. Now, let $V \in \mathcal{V}_\rho \cap \mathbb{X}_2$. Writing $V = V - C + C$ with $C = (c, c)$,

$$c := \frac{1}{(\lambda_1 + \lambda_2)} \left(\int_{\Omega} v \, dx + \int_{\partial\Omega} v \frac{d\sigma}{b} \right),$$

and using the fact that $\int_{\Omega} F \, d\mu = 0$, we obtain, for every $V \in \mathcal{V}_\rho \cap \mathbb{X}_2$, that

$$\int_{\Omega} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla v \, dx + \rho \int_{\partial\Omega} |\nabla_\Gamma u^*|^{q-2} \nabla_\Gamma u^* \cdot \nabla_\Gamma v \, d\sigma = \int_{\Omega} FV \, d\mu.$$

Therefore, $A_\rho(U) = F$. Hence, $F \in \mathcal{R}(A_\rho)$ and this completes the proof of the lemma. \square

The following result is a direct consequence of Lemmas 3.3, 3.4. This is the main result of this section.

Theorem 3.5. *The operator $A_\rho = \partial \mathcal{J}_\rho$ satisfies the following type of "quasi-linear" Fredholm alternative:*

$$\mathcal{R}(A_\rho) = \mathcal{N}(A_\rho)^\perp = \left\{ F \in \mathbb{X}_2 : \langle F, \mathbf{1} \rangle_{\mathbb{X}_2} = 0 \right\}.$$

4. NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE OF SOLUTIONS

In this section, we prove the first main result (cf. Theorem 1.1) for problem (1.1). Before we do so, we will need the following results from maximal monotone operators theory and convex analysis.

Definition 4.1. *Let \mathcal{H} be a real Hilbert space. Two subsets K_1 and K_2 of \mathcal{H} are said to be almost equal, written, $K_1 \simeq K_2$, if K_1 and K_2 have the same closure and the same interior, that is, $\overline{K_1} = \overline{K_2}$ and $\text{int}(K_1) = \text{int}(K_2)$.*

The following abstract result is taken from [8, Theorem 3 and Generalization in p.173–174].

Theorem 4.2 (Brezis-Haraux). *Let A and B be subdifferentials of proper convex lower semicontinuous functionals φ_1 and φ_2 , respectively, on a real Hilbert space \mathcal{H} with $D(\varphi_1) \cap D(\varphi_2) \neq \emptyset$, and let C be the subdifferential of the proper, convex lower semicontinuous functional $\varphi_1 + \varphi_2$, that is $C = \partial(\varphi_1 + \varphi_2)$. Then*

$$\mathcal{R}(A) + \mathcal{R}(B) \subset \overline{\mathcal{R}(C)} \quad \text{and} \quad \text{Int}(\mathcal{R}(A) + \mathcal{R}(B)) \subset \mathcal{R}(C)$$

In particular, if the operator $A + B$ is maximal monotone, then

$$\mathcal{R}(A + B) \simeq \mathcal{R}(A) + \mathcal{R}(B),$$

and this is the case if $\partial(\varphi_1 + \varphi_2) = \partial\varphi_1 + \partial\varphi_2$.

4.1. Assumptions and intermediate results. Let us recall that the aim of this section is to establish some necessary and sufficient conditions for the solvability of the following nonlinear elliptic problem:

$$\begin{cases} -\Delta_p u + \alpha_1(u) = f, & \text{in } \Omega, \\ b(x) |\nabla u|^{p-2} \partial_n u - \rho b(x) \Delta_{q,\Gamma} u + \alpha_2(u) = g, & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where $p, q \in (1, +\infty)$ are fixed. We also assume that $\alpha_j : \mathbb{R} \rightarrow \mathbb{R}$ ($j = 1, 2$) satisfy the following assumptions.

Assumption 4.3. *The functions $\alpha_j : \mathbb{R} \rightarrow \mathbb{R}$ ($j = 1, 2$) are odd, monotone nondecreasing, continuous and satisfy $\alpha_j(0) = 0$.*

Let $\tilde{\alpha}_j$ be the inverse of α_j . We define the functions $\Lambda_j, \tilde{\Lambda}_j : \mathbb{R} \rightarrow \mathbb{R}_+$ ($j = 1, 2$) by

$$\Lambda_j(t) := \int_0^{|t|} \alpha_j(s) ds \quad \text{and} \quad \tilde{\Lambda}_j(t) := \int_0^{|t|} \tilde{\alpha}_j(s) ds. \quad (4.2)$$

Then it is clear that $\Lambda_j, \tilde{\Lambda}_j$ are even, convex and monotone increasing on \mathbb{R}_+ , with $\Lambda_j(0) = \tilde{\Lambda}_j(0)$, for each $j = 1, 2$. Moreover, since α_j are odd, we have $\Lambda_j'(t) = \alpha_j(t)$, for all $t \in \mathbb{R}$ and $j = 1, 2$, with a similar relation holding for $\tilde{\Lambda}_j$ as well. The following result whose proof is included in [25, Chap. I, Section 1.3, Theorem 3] holds.

Lemma 4.4. *The functions Λ_j and $\tilde{\Lambda}_j$ ($j = 1, 2$) satisfy (2.6) and (2.7). More precisely, for all $s, t \in \mathbb{R}$,*

$$st \leq \Lambda_j(s) + \tilde{\Lambda}_j(t).$$

If $s = \alpha_j(t)$ or $t = \tilde{\alpha}_j(s)$, then we also have equality, that is,

$$\tilde{\Lambda}_j(\alpha_j(s)) = s\alpha_j(s) - \Lambda_j(s), \quad j = 1, 2.$$

We note that in [25], the statement of Lemma 4.4 assumed that $\Lambda_j, \tilde{\Lambda}_j$ are \mathcal{N} -functions in the sense of Definition 2.5. However, the conclusion of that result holds under the weaker hypotheses of Lemma 4.4.

Define the functional $\mathcal{J}_2 : \mathbb{X}_2 \rightarrow [0, +\infty]$ by

$$\mathcal{J}_2(u, v) := \begin{cases} \int_{\Omega} \Lambda_1(u) dx + \int_{\partial\Omega} \Lambda_2(v) \frac{d\sigma}{b}, & \text{if } (u, v) \in D(\mathcal{J}_2), \\ +\infty, & \text{if } (u, v) \in \mathbb{X}_2 \setminus D(\mathcal{J}_2), \end{cases}$$

with the effective domain

$$D(\mathcal{J}_2) := \left\{ (u, v) \in \mathbb{X}_2 : \int_{\Omega} \Lambda_1(u) dx + \int_{\partial\Omega} \Lambda_2(v) \frac{d\sigma}{b} < \infty \right\}.$$

Lemma 4.5. *Let α_j ($j = 1, 2$) satisfy Assumption 4.3. Then the functional \mathcal{J}_2 is proper, convex and lower semicontinuous on \mathbb{X}_2 .*

Proof. It is routine to check that \mathcal{J}_2 is convex and proper. This follows easily from the convexity of Λ_j and the fact that $\Lambda_j(0) = 0$. To show the lower semicontinuity on \mathbb{X}_2 , let $U_n = (u_n, v_n) \in D(\mathcal{J}_2)$ be such that $U_n \rightarrow U := (u, v)$ in \mathbb{X}_2 and $\mathcal{J}_2(U_n) \leq C$ for some constant $C > 0$. Since $U_n \rightarrow U$ in \mathbb{X}_2 , then there is a subsequence, which we also denote by $U_n = (u_n, v_n)$, such that $u_n \rightarrow u$ a.e. on Ω and $v_n \rightarrow v$ σ -a.e. on Γ . Since $\Lambda_j(\cdot)$ are continuous (thus, lower-semicontinuous), we have

$$\Lambda_1(u) \leq \liminf_{n \rightarrow \infty} \Lambda_1(u_n) \quad \text{and} \quad \Lambda_2(v) \leq \liminf_{n \rightarrow \infty} \Lambda_2(v_n).$$

By Fatou's Lemma, we obtain

$$\int_{\Omega} \Lambda_1(u) dx \leq \int_{\Omega} \liminf_{n \rightarrow \infty} \Lambda_1(u_n) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \Lambda_1(u_n) dx$$

and

$$\int_{\partial\Omega} \Lambda_2(v) \frac{d\sigma}{b} \leq \int_{\partial\Omega} \liminf_{n \rightarrow \infty} \Lambda_2(v_n) \frac{d\sigma}{b} \leq \liminf_{n \rightarrow \infty} \int_{\partial\Omega} \Lambda_2(v_n) \frac{d\sigma}{b}.$$

Hence, \mathcal{J}_2 is lower semicontinuous on \mathbb{X}_2 . \square

We have the following result whose proof is contained in [25, Chap. III, Section 3.1, Theorem 2].

Lemma 4.6. *Let α_j ($j = 1, 2$) satisfy Assumption 4.3 and assume that there exist constants $C_j > 1$ ($j = 1, 2$) such that*

$$\Lambda_j(2t) \leq C_j \Lambda_j(t), \text{ for all } t \in \mathbb{R}. \quad (4.3)$$

Then $D(\mathcal{J}_2)$ is a vector space.

Let the operator B_2 be defined by

$$\begin{cases} D(B_2) = \{U := (u, v) \in \mathbb{X}_2 : (\alpha_1(u), \alpha_2(v)) \in \mathbb{X}_2\}, \\ B_2(U) = (\alpha_1(u), \alpha_2(v)). \end{cases} \quad (4.4)$$

We have the following result.

Lemma 4.7. *Let the assumptions of Lemma 4.6 be satisfied. Then the subdifferential of \mathcal{J}_2 and the operator B_2 coincide, that is, for all $(u, v) \in D(B_2) = D(\partial \mathcal{J}_2)$,*

$$\partial \mathcal{J}_2(u, v) = B_2(u, v).$$

Proof. Let $U = (u, v) \in D(\mathcal{J}_2)$ and $F = (f, g) \in \partial \mathcal{J}_2(u, v)$. Then by definition, $F \in \mathbb{X}_2$ and, for every $V = (u_1, v_1) \in D(\mathcal{J}_2)$, we get

$$\int_{\Omega} F(V - U) d\mu \leq \mathcal{J}_2(V) - \mathcal{J}_2(U).$$

Let $V = U + tW$, with $W = (u_2, v_2) \in D(\mathcal{J}_2)$ and $0 < t \leq 1$. Then by Lemma 4.6, $V = U + tW \in D(\mathcal{J}_2)$. Now, dividing by t and taking the limit as $t \downarrow 0$, we obtain

$$\int_{\Omega} FW d\mu \leq \int_{\Omega} \alpha_1(u) u_2 dx + \int_{\partial\Omega} \alpha_2(v) v_2 \frac{d\sigma}{b}. \quad (4.5)$$

Changing W to $-W$ in (4.5) gives that

$$\int_{\Omega} FW d\mu = \int_{\Omega} \alpha_1(u) u_2 dx + \int_{\partial\Omega} \alpha_2(v) v_2 \frac{d\sigma}{b}.$$

In particular, if $W = (u_2, 0)$ with $u_2 \in \mathcal{D}(\Omega)$, we have

$$\int_{\Omega} f u_2 dx = \int_{\Omega} \alpha_1(u) u_2 dx,$$

and this shows that $\alpha_1(u) = f$. Similarly, one obtains that $\alpha_2(v) = g$. We have shown that $U \in D(B_2)$ and

$$B_2(U) := B_2(u, v) = (\alpha_1(u), \alpha_2(v)) = (f, g).$$

Conversely, let $U = (u, v) \in D(B_2)$ and set $F = (f, g) := B_2(u, v) = (\alpha_1(u), \alpha_2(v))$. Since $(\alpha_1(u), \alpha_2(v)) \in \mathbb{X}_2$, from (4.2) and (4.3), it follows that

$$\int_{\Omega} \Lambda_1(u) dx + \int_{\partial\Omega} \Lambda_2(v) \frac{d\sigma}{b} < \infty.$$

Hence, $U = (u, v) \in D(\mathcal{J}_2)$. Let $V = (u_1, v_1) \in D(\mathcal{J}_2)$. Using Lemma 4.4, we obtain

$$\begin{aligned}\alpha_1(u)(u_1 - u) &= \alpha_1(u)u_1 - \alpha_1(u)u \\ &\leq \Lambda_1(u_1) + \Lambda_1(\alpha_1(u)) - \alpha_1(u)u \\ &= \Lambda_1(u_1) - \Lambda_1(u)\end{aligned}\quad (4.6)$$

and similarly,

$$\alpha_2(v)(v_1 - v) \leq \Lambda_2(v_1) - \Lambda_2(v).$$

Therefore,

$$\begin{aligned}\int_{\Omega} F(V - U) d\mu &= \int_{\Omega} \alpha_1(u)(u_1 - u) dx + \int_{\partial\Omega} \alpha_2(v)(v_1 - v) \frac{d\sigma}{b} \\ &\leq \mathcal{J}_2(V) - \mathcal{J}_2(U).\end{aligned}$$

By definition, this shows that $F = (\alpha_1(u), \alpha_2(v)) = B_2(U) \in \partial \mathcal{J}_2(U)$. We have shown that $U \in D(\partial \mathcal{J}_2)$ and $B_2(U) \in \partial \mathcal{J}_2(U)$. This completes the proof of the lemma. \square

Next, we define the functional $\mathcal{J}_{3,\rho} : \mathbb{X}_2 \rightarrow [0, +\infty]$ by

$$\mathcal{J}_{3,\rho}(U) = \begin{cases} \mathcal{J}_{\rho}(U) + \mathcal{J}_2(U) & \text{if } U \in D(\mathcal{J}_{3,\rho}) := D(\mathcal{J}_{\rho}) \cap D(\mathcal{J}_2), \\ +\infty & \text{if } U \in \mathbb{X}_2 \setminus D(\mathcal{J}_{3,\rho}). \end{cases} \quad (4.7)$$

Note that for $\rho = 0$,

$$D(\mathcal{J}_{3,0}) = \{U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_2) : u \in W^{1,p}(\Omega) \cap L^2(\Omega), u|_{\partial\Omega} \in L^2(\partial\Omega)\}, \quad (4.8)$$

while for $\rho = 1$,

$$D(\mathcal{J}_{3,1}) = \{U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_2) : u \in W^{1,p}(\Omega) \cap L^2(\Omega), u|_{\partial\Omega} \in W^{1,q}(\partial\Omega) \cap L^2(\partial\Omega)\}. \quad (4.9)$$

We have the following result.

Lemma 4.8. *Let the assumptions of Lemma 4.6 be satisfied. Then the subdifferential of the functional $\mathcal{J}_{3,\rho}$ is given by*

$$\begin{aligned}D(\partial \mathcal{J}_{3,\rho}) &= \{U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_{3,\rho}) : -\Delta_p u + \alpha_1(u) \in L^2(\Omega) \\ &\quad \text{and } b(x)|\nabla u|^{p-2}\partial_n u - b(x)\rho\Delta_{q,\Gamma}u + \alpha_2(u) \in L^2(\partial\Omega, d\sigma/b)\}\end{aligned}$$

and

$$\partial \mathcal{J}_{3,\rho}(U) = \left(-\Delta_p u + \alpha_1(u), b(x)|\nabla u|^{p-2}\partial_n u - b(x)\rho\Delta_{q,\Gamma}u + \alpha_2(u) \right). \quad (4.10)$$

In particular, if for every $U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_{3,\rho})$, the function $(\alpha_1(u), \alpha_2(u)) \in \mathbb{X}_2$, then

$$\partial \mathcal{J}_{3,\rho} := \partial(\mathcal{J}_{\rho} + \mathcal{J}_2) = \partial \mathcal{J}_{\rho} + \partial \mathcal{J}_2.$$

Proof. We calculate the subdifferential $\partial \mathcal{J}_{3,\rho}$. Let $F = (f, g) \in \partial \mathcal{J}_{3,\rho}(U)$, that is, $F \in \mathbb{X}_2$, $U \in D(\mathcal{J}_{3,\rho}) = D(\mathcal{J}_{\rho}) \cap D(\mathcal{J}_2)$ and for every $V \in D(\mathcal{J}_{3,\rho})$, we have

$$\int_{\Omega} F(V - U) d\mu \leq \mathcal{J}_{3,\rho}(V) - \mathcal{J}_{3,\rho}(U).$$

Proceeding as in Remark 3.2 and the proof of Lemma 4.7, we obtain that

$$-\Delta_p u + \alpha_1(u) = f \quad \text{in } \mathcal{D}(\Omega)',$$

and

$$b(x)|\nabla u|^{p-2}\partial_n u - b(x)\rho\Delta_{q,\Gamma}u + \alpha_2(u) = g \quad \text{weakly on } \partial\Omega.$$

Noting that $\partial \mathcal{J}_{3,\rho}$ is also a single-valued operator (which follows from the assumptions on α_j and Λ_j), we easily obtain (4.10), and this completes the proof of the first part.

To show the last part, note that it is clear that $\partial \mathcal{J}_\rho + \partial \mathcal{J}_2 \subset \partial \mathcal{J}_{3,\rho}$ always holds. To show the converse inclusion, let assume that for every $U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_{3,\rho})$, the function $(\alpha_1(u), \alpha_2(u)) \in \mathbb{X}_2$. Then it follows from (3.3), (4.4) (since $\partial \mathcal{J}_2 = B_2$) and (4.10), that $D(\partial \mathcal{J}_{3,\rho}) = D(\partial \mathcal{J}_\rho) \cap D(\partial \mathcal{J}_2)$ and

$$\begin{aligned} \partial \mathcal{J}_{3,\rho}(U) &= (-\Delta_p u + \alpha_1(u), b(x)|\nabla u|^{p-2}\partial_n u - b(x)\rho\Delta_{q,\Gamma}u + \alpha_2(u)) \\ &= (-\Delta_p u, b(x)|\nabla u|^{p-2}\partial_n u - b(x)\rho\Delta_{q,\Gamma}u) + (\alpha_1(u), \alpha_2(u)) \\ &= \partial \mathcal{J}_\rho(U) + \partial \mathcal{J}_2(U). \end{aligned}$$

This completes the proof. \square

The following lemma is the main ingredient in the proof of Theorem 4.11 below.

Lemma 4.9. *Let $B_1 := A_\rho$ and set $B_3 := \partial \mathcal{J}_{3,\rho}$. Then*

$$\mathcal{R}(B_1) + \mathcal{R}(B_2) \subset \overline{\mathcal{R}(B_3)} \text{ and } \text{Int}(\mathcal{R}(B_1) + \mathcal{R}(B_2)) \subset \mathcal{R}(B_3). \quad (4.11)$$

In particular, if for every $U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_{3,\rho})$, the function $(\alpha_1(u), \alpha_2(u)) \in \mathbb{X}_2$, then

$$\mathcal{R}(B_3) := \mathcal{R}(B_1 + B_2) \simeq \mathcal{R}(B_1) + \mathcal{R}(B_2). \quad (4.12)$$

Proof. By Remark 3.2 and Lemmas 4.7, 4.8, the operators B_1 , B_2 and B_3 are subdifferentials of proper, convex and lower semicontinuous functionals \mathcal{J}_ρ , \mathcal{J}_2 and $\mathcal{J}_\rho + \mathcal{J}_2$, respectively, on \mathbb{X}_2 . Hence, B_1 , B_2 and B_3 are maximal monotone operators. In particular, if $(\alpha_1(u), \alpha_2(u)) \in \mathbb{X}_2$, for every $U = (u, u|_{\partial\Omega}) \in D(\mathcal{J}_{3,\rho})$, then by Lemma 4.8, one has $B_3 = B_1 + B_2$. Now, the lemma follows from the celebrated Brezis-Haraux result in Theorem 4.2. \square

4.2. Statement and proof of the main result. Next, let $\mathcal{V}_\rho := D(\mathcal{J}_{3,\rho})$ be given by (4.8) if $\rho = 0$ and by (4.9) if $\rho = 1$.

Definition 4.10. *Let $F = (f, g) \in \mathbb{X}_2$. A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (4.1), if $\alpha_1(u) \in L^1(\Omega)$, $\alpha_2(u) \in L^1(\partial\Omega)$, $u|_{\partial\Omega} \in W^{1,q}(\partial\Omega)$, if $\rho > 0$ and*

$$\begin{aligned} &\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \rho \int_{\partial\Omega} |\nabla_{\Gamma} u|^{q-2} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v d\sigma \\ &+ \int_{\Omega} \alpha_1(u) v dx + \int_{\partial\Omega} \alpha_2(u) v \frac{d\sigma}{b} = \int_{\Omega} f v dx + \int_{\partial\Omega} g v \frac{d\sigma}{b}, \end{aligned} \quad (4.13)$$

for every $v \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $v|_{\partial\Omega} \in W^{1,q}(\partial\Omega)$, if $\rho > 0$.

Recall that $\lambda_1 := \int_{\Omega} dx$ and $\lambda_2 := \int_{\partial\Omega} \frac{d\sigma}{b}$. We also define the average $\langle F \rangle_{\overline{\Omega}}$ of $F = (f, g)$ with respect to the measure μ , as follows:

$$\langle F \rangle_{\overline{\Omega}} := \frac{1}{\mu(\overline{\Omega})} \int_{\overline{\Omega}} F d\mu = \frac{1}{\mu(\overline{\Omega})} \left(\int_{\Omega} f dx + \int_{\partial\Omega} g \frac{d\sigma}{b} \right),$$

where $\mu(\overline{\Omega}) = \lambda_1 + \lambda_2$. Now, we are ready to state the main result of this section.

Theorem 4.11. *Let α_j ($j = 1, 2$) satisfy Assumption 4.3 and assume that the functions Λ_j ($j = 1, 2$) satisfy (4.3). Let $F = (f, g) \in \mathbb{X}_2$. The following hold:*

(a) *Suppose that the nonlinear elliptic problem (4.1) possesses a weak solution. Then*

$$\langle F \rangle_{\overline{\Omega}} \in \frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2}. \quad (4.14)$$

(b) Assume that

$$\langle F \rangle_{\overline{\Omega}} \in \text{int} \left(\frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right). \quad (4.15)$$

Then the nonlinear elliptic problem (4.1) has at least one weak solution.

Proof. We show that condition (4.14) is necessary. Let $F := (f, g) \in \mathbb{X}_2$ and let $U = (u, u|_{\partial\Omega}) \in D(B_3) \subset \mathcal{V}_\rho$ be a weak solution of $B_3 U = F$. Then, by definition, for every $V = (v, v|_{\partial\Omega}) \in \mathcal{V}_\rho$, (4.13) holds. Taking $v \equiv 1$ in (4.13) yields

$$\int_{\Omega} f \, dx + \int_{\partial\Omega} g \frac{d\sigma}{b} = \int_{\Omega} \alpha_1(u) \, dx + \int_{\partial\Omega} \alpha_2(u) \frac{d\sigma}{b}.$$

Hence,

$$\int_{\Omega} f \, dx + \int_{\partial\Omega} g \frac{d\sigma}{b} \in (\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)),$$

and so (4.14) holds. This completes the proof of part (a).

We show that the condition (4.15) is sufficient.

(i) First, let $C \in \mathbf{C}$, where

$$\mathbf{C} := \{C = (c_1, c_2) : (c_1, c_2) \in \mathcal{R}(\alpha_1) \times \mathcal{R}(\alpha_2)\}.$$

By definition, one has that $\mathbf{C} \subset \mathcal{R}(B_2)$ since $c_1 = \alpha_1(d_1)$ for some constant function d_1 on Ω and $c_2 = \alpha_2(d_2)$ for some constant function d_2 on $\partial\Omega$. Let $F \in \mathbb{X}_2$ be such that (4.15) holds. We must show $F \in \mathcal{R}(B_3)$. By (4.15), we may choose $C = (c_1, c_2) \in \mathbf{C}$ such that

$$\langle F \rangle_{\overline{\Omega}} = \frac{\lambda_1 c_1 + \lambda_2 c_2}{\lambda_1 + \lambda_2} \in \text{int} \left(\frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right).$$

Then, for $F \in \mathbb{X}_2$, we have $F = F_1 + F_2$ with

$$F_1 := F - C \text{ and } F_2 = C.$$

First, $F_1 \in \mathcal{R}(B_1) = \mathcal{N}(B_1)^\perp = \mathbf{1}^\perp$, since

$$\begin{aligned} \int_{\Omega} F_1 \, d\mu &= \int_{\Omega} (F - C) \, d\mu \\ &= \int_{\Omega} f \, dx + \int_{\partial\Omega} g \frac{d\sigma}{b} - (\lambda_1 c_1 + \lambda_2 c_2) \\ &= (\lambda_1 + \lambda_2) \langle F \rangle_{\overline{\Omega}} - (\lambda_1 c_1 + \lambda_2 c_2) = 0. \end{aligned}$$

Obviously, $F_2 = C \in \mathcal{R}(B_2)$. Hence, it is readily seen that

$$F \in (\mathcal{R}(B_1) + \mathcal{R}(B_2)).$$

(ii) Next, denote by $\mathbb{B}_{\mathbb{R}}(x, r)$ the open ball in \mathbb{R} of center x and radius $r > 0$. Since

$$\langle F \rangle_{\overline{\Omega}} \in \text{int} \left(\frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right),$$

there exists $\delta > 0$ such that the open ball

$$\mathbb{B}_{\mathbb{R}}(\langle F \rangle_{\overline{\Omega}}, \delta) \subset \left(\frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right).$$

Since the mapping $F \mapsto \langle F \rangle_{\overline{\Omega}}$ from \mathbb{X}_2 into \mathbb{R} is continuous, then there exists $\varepsilon > 0$ such that

$$\langle G \rangle_{\overline{\Omega}} \in \mathbb{B}_{\mathbb{R}}(\langle F \rangle_{\overline{\Omega}}, \delta) \subset \left(\frac{\lambda_1 \mathcal{R}(\alpha_1) + \lambda_2 \mathcal{R}(\alpha_2)}{\lambda_1 + \lambda_2} \right),$$

for all $G \in \mathbb{X}_2$ satisfying $\|F - G\|_2 < \varepsilon$. It finally follows from part (i) above that $(\mathcal{R}(B_1) + \mathcal{R}(B_2))$ contains an ε -ball in \mathbb{X}_2 centered at F . Therefore,

$$F \in \text{int}(\mathcal{R}(B_1) + \mathcal{R}(B_2)) \subset \mathcal{R}(B_3).$$

Consequently, problem (4.1) is (weakly) solvable for every function $F = (f, g) \in \mathbb{X}_2$, if (4.15) holds. This completes the proof of the theorem. \square

Remark 4.12. *It is important to remark that in order to prove Theorem 4.11, we do not require that $(\alpha_1(u), \alpha_2(u))$ should belong to \mathbb{X}_2 , for every $U = (u, u|_\Gamma) \in D(\mathcal{J}_{3,p})$. In particular, only the assumption (4.11) was needed. However, if this happens, then we get the much stronger result in (4.12) which would require that the nonlinearities α_1, α_2 satisfy growth assumptions at infinity.*

We conclude this section with the following corollary and some examples.

Corollary 4.13. *Let the assumptions of Theorem 4.11 be satisfied. Let $F = (f, g) \in \mathbb{X}_2$. Assume that at least one of the sets $\mathcal{R}(\alpha_1)$, $\mathcal{R}(\alpha_2)$ is open. Then the nonlinear elliptic problem (4.1) possesses a weak solution if and only if (4.15) holds.*

Remark 4.14. *Similar results to Theorem 4.11 and Corollary 4.13 were also obtained in [12, Theorem 4.4], but only when $p = q = 2$.*

4.3. Examples. We will now give some examples as applications of Theorem 4.11. Let $p, q \in (1, +\infty)$ be fixed.

Example 4.15. *Let $\alpha_1(s)$ or $\alpha_2(s)$ be equal to $\alpha(s) = c|s|^{r-1}s$, where $c, r > 0$. Note that $\mathcal{R}(\alpha) = \mathbb{R}$. It is easy to check that α satisfies all the conditions of Assumption 4.3 and that the function $\Lambda(t) = \int_0^{|t|} \alpha(s)ds$ satisfies (4.3). Then, it follows that problem (4.1) is solvable for any $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$.*

Example 4.16. *Consider the case when $p = \alpha_2 \equiv 0$ in (4.1), that is, consider the following boundary value problem:*

$$\begin{cases} -\Delta_p u + \alpha_1(u) = f \text{ in } \Omega, \\ b(x)|\nabla u|^{p-2} \partial_n u = g \text{ on } \Gamma. \end{cases}$$

Then, by Theorem 4.11, this problem has a weak solution if

$$\int_{\Omega} f \, dx + \int_{\partial\Omega} g \, \frac{d\sigma}{b} \in \lambda_1 \text{int}(\mathcal{R}(\alpha_1)),$$

which yields the classical Landesman-Lazer result (see (1.6)) for $g \equiv 0$ and $p = 2$.

Example 4.17. *Let us now consider the case when $\alpha_1 \equiv \alpha$ and $\alpha_2 \equiv 0$, where α is a continuous, odd and nondecreasing function on \mathbb{R} such that $\alpha(0) = 0$. The problem*

$$\begin{cases} -\Delta_p u + \alpha(u) = f, & \text{in } \Omega, \\ b(x)|\nabla u|^{p-2} \partial_n u - \rho b(x) \Delta_{q,\Gamma} u = g, & \text{on } \partial\Omega, \end{cases} \quad (4.16)$$

has a weak solution if

$$\int_{\Omega} f \, dx + \int_{\partial\Omega} g \, \frac{d\sigma}{b} \in \lambda_2 \text{int}(\mathcal{R}(\alpha)). \quad (4.17)$$

Let us now choose $\alpha(s) = \arctan(s)$ in (4.16). Then, it is easy to check that

$$\Lambda(t) := \int_0^{|t|} \alpha(s)ds = |t| \arctan(|t|) - \frac{1}{2} \ln(1+t^2), \quad t \in \mathbb{R}$$

is monotone increasing on \mathbb{R}_+ and that it satisfies $\Lambda(2t) \leq C_2\Lambda(t)$, $\forall t \in \mathbb{R}$, for some constant $C_2 > 1$. Therefore, (4.17) becomes the necessary and sufficient condition

$$\left| \frac{1}{\lambda_2} \left(\int_{\Omega} f \, dx + \int_{\partial\Omega} g \frac{d\sigma}{b} \right) \right| < \frac{\pi}{2}. \quad (4.18)$$

5. A PRIORI ESTIMATES

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain with boundary $\partial\Omega$. Recall that $1 < p, q < \infty$, $\rho \in \{0, 1\}$ and $b \in L^\infty(\partial\Omega)$ with $b(x) \geq b_0 > 0$, for some constant b_0 . We consider the nonlinear elliptic boundary value problem formally given by

$$\begin{cases} -\Delta_p u + \alpha_1(x, u) + |u|^{p-2}u = f, & \text{in } \Omega \\ -\rho b(x)\Delta_{q,\Gamma} u + \rho b(x)|u|^{q-2}u + b(x)|\nabla u|^{p-2}\partial_n u + \alpha_2(x, u) = g, & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where $f \in L^{p_1}(\Omega)$ and $g \in L^{q_1}(\partial\Omega)$ for some $1 \leq p_1, q_1 \leq \infty$. If $\rho = 0$, then the boundary conditions in (5.1) are of Robin type. Existence and regularity of weak solutions for this case have been obtained in [5] for $p = 2$ (see also [29] for the linear case) and for general p in [6]. Therefore, we will concentrate our attention to the case $\rho = 1$ only; in this case, the boundary condition in (5.1) is a generalized Wentzell-Robin boundary condition. For the sake of simplicity, from now on we will also take $b \equiv 1$.

5.1. General assumptions. Throughout this section, we assume that the functions $\alpha_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $\alpha_2 : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following conditions:

Assumption 5.1.

$$\begin{cases} \alpha_j(x, \cdot) \text{ is odd and strictly increasing,} \\ \alpha_j(x, 0) = 0, \quad \alpha_j(x, \cdot) \text{ is continuous,} \\ \lim_{t \rightarrow 0} \frac{\alpha_j(x, t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{\alpha_j(x, t)}{t} = \infty, \end{cases}$$

for λ_N -a.e. $x \in \Omega$ if $j = 1$ and σ -a.e. $x \in \partial\Omega$ if $j = 2$.

Since $\alpha_j(x, \cdot)$ are strictly increasing for λ_N -a.e. $x \in \Omega$ if $j = 1$ and σ -a.e. $x \in \partial\Omega$ if $j = 2$, then they have inverses which we denote by $\tilde{\alpha}_j(x, \cdot)$ (cf. also Section 4). We define the functions $\Lambda_1, \tilde{\Lambda}_1 : \Omega \times \mathbb{R} \rightarrow [0, \infty)$ and $\Lambda_2, \tilde{\Lambda}_2 : \partial\Omega \times \mathbb{R} \rightarrow [0, \infty)$ by

$$\Lambda_j(x, t) := \int_0^{|t|} \alpha_j(x, s) \, ds \quad \text{and} \quad \tilde{\Lambda}_j(x, t) := \int_0^{|t|} \tilde{\alpha}_j(x, s) \, ds.$$

Then, it is clear that, for λ_N -a.e. $x \in \Omega$ if $j = 1$ and σ -a.e. $x \in \partial\Omega$ if $j = 2$, $\Lambda_j(x, \cdot)$ and $\tilde{\Lambda}_j(x, \cdot)$ are differentiable, monotone and convex with $\Lambda_j(x, 0) = \tilde{\Lambda}_j(x, 0) = 0$. Furthermore, $\Lambda_j(x, \cdot)$ is an \mathcal{N} -function and $\tilde{\Lambda}_j(x, \cdot)$ is its complementary \mathcal{N} -function. The function $\tilde{\Lambda}_j$ is then the complementary Musielak-Orlick function of Λ_j in the sense of Young (see Definition 2.3).

Assumption 5.2. We assume, for λ_N -a.e. $x \in \Omega$ if $j = 1$ and σ -a.e. $x \in \partial\Omega$ if $j = 2$, that $\Lambda_j(x, \cdot)$ and $\tilde{\Lambda}_j(x, \cdot)$ satisfy the (Δ_2) -condition in the sense of Definition 2.5.

It follows from Assumption 5.2 that there exist two constants $c_1, c_2 \in (0, 1]$ such that for λ_N -a.e. $x \in \Omega$ if $j = 1$ and σ -a.e. $x \in \partial\Omega$ if $j = 2$ and for all $t \in \mathbb{R}$,

$$c_j t \alpha_j(x, t) \leq \Lambda_j(x, t) \leq t \alpha_j(x, t). \quad (5.2)$$

Next, let

$$L_{\Lambda_1}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable: } \int_{\Omega} \Lambda_1(x, u) dx < \infty \right\}$$

and

$$L_{\Lambda_2}(\partial\Omega) := \left\{ u : \partial\Omega \rightarrow \mathbb{R} \text{ measurable: } \int_{\partial\Omega} \Lambda_2(x, u) d\sigma < \infty \right\}.$$

Since $\Lambda_j(x, \cdot)$ and $\tilde{\Lambda}_j(x, \cdot)$ satisfy the (Δ_2) -condition, it follows from [1, Theorem 8.19], that $L_{\Lambda_1}(\Omega)$ and $L_{\Lambda_2}(\partial\Omega)$, endowed respectively with the norms

$$\|u\|_{\Lambda_1, \Omega} := \inf \left\{ k > 0 : \int_{\Omega} \Lambda_1 \left(x, \frac{u(x)}{k} \right) dx \leq 1 \right\},$$

and

$$\|u\|_{\Lambda_2, \partial\Omega} := \inf \left\{ k > 0 : \int_{\partial\Omega} \Lambda_2 \left(x, \frac{u(x)}{k} \right) d\sigma \leq 1 \right\},$$

are reflexive Banach spaces. Moreover, by [1, Section 8.11, p.234], the following generalized versions of Hölder's inequality will also become useful in the sequel,

$$\left| \int_{\Omega} uv dx \right| \leq 2 \|u\|_{\Lambda_1, \Omega} \|v\|_{\tilde{\Lambda}_1, \Omega} \quad (5.3)$$

and

$$\left| \int_{\partial\Omega} uv d\sigma \right| \leq 2 \|u\|_{\Lambda_2, \partial\Omega} \|v\|_{\tilde{\Lambda}_2, \partial\Omega}. \quad (5.4)$$

5.2. Existence and uniqueness of weak solutions of perturbed equations. Let

$$\mathcal{V} := \{U := (u, u|_{\partial\Omega}) : u \in W^{1,p}(\Omega) \cap L_{\Lambda_1}(\Omega), u|_{\partial\Omega} \in W^{1,q}(\partial\Omega) \cap L_{\Lambda_2}(\partial\Omega)\}.$$

Then for every $1 < p, q < \infty$, \mathcal{V} endowed with the norm

$$\|U\|_{\mathcal{V}} = \|u\|_{W^{1,p}(\Omega)} + \|u\|_{\Lambda_1, \Omega} + \|u\|_{W^{1,q}(\partial\Omega)} + \|u\|_{\Lambda_2, \partial\Omega}$$

is a reflexive Banach space. Recall that $p = 1$. Throughout the following, we denote by \mathcal{V}' the dual of \mathcal{V} .

Definition 5.3. A function $U = (u, u|_{\partial\Omega}) \in \mathcal{V}$ is said to be a weak solution of (5.1), if for every $V \in \mathcal{V} = (v, v|_{\partial\Omega})$,

$$\mathcal{A}(U, V) = \int_{\Omega} f v dx + \int_{\partial\Omega} g v d\sigma, \quad (5.5)$$

provided that the integrals on the right-hand side exist. Here,

$$\begin{aligned} \mathcal{A}(U, V) &:= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_{\Omega} |u|^{p-2} u v dx \\ &\quad + \int_{\Omega} \alpha_1(x, u) v dx + \int_{\partial\Omega} |\nabla_{\Gamma} u|^{q-2} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v d\sigma \\ &\quad + \int_{\partial\Omega} |u|^{q-2} u v d\sigma + \int_{\partial\Omega} \alpha_2(x, u) v d\sigma. \end{aligned}$$

Lemma 5.4. Assume Assumptions 5.1 and 5.2. Let $1 < p, q < \infty$ and $U \in \mathcal{V}$ be fixed. Then the functional $V \mapsto \mathcal{A}(U, V)$ belongs to \mathcal{V}' . Moreover, \mathcal{A} is strictly monotone, hemicontinuous and coercive.

Proof. Let $U = (u, u|_{\partial\Omega}) \in \mathcal{V}$ be fixed. It is clear that $\mathcal{A}(U, \cdot)$ is linear. Let $V = (v, v|_{\partial\Omega}) \in \mathcal{V}$. Then, exploiting (5.3) and (5.4), we obtain

$$\begin{aligned} |\mathcal{A}(U, V)| &\leq \|u\|_{W^{1,p}(\Omega)}^{p-1} \|v\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,q}(\partial\Omega)}^{q-1} \|v\|_{W^{1,q}(\partial\Omega)} \\ &\quad + 2 \max \left\{ 1, \int_{\Omega} \tilde{\Lambda}_1(x, \alpha_1(x, u)) dx \right\} \|v\|_{\Lambda_1, \Omega} \\ &\quad + 2 \max \left\{ 1, \int_{\partial\Omega} \tilde{\Lambda}_2(x, \alpha_2(x, u)) d\sigma \right\} \|v\|_{\Lambda_2, \partial\Omega} \\ &\leq K(U) \|V\|_{\mathcal{V}}, \end{aligned} \quad (5.6)$$

where

$$\begin{aligned} K(U) &:= \|u\|_{W^{1,p}(\Omega)}^{p-1} + 2 \max \left\{ 1, \int_{\Omega} \tilde{\Lambda}_1(x, \alpha_1(x, u)) dx \right\} \\ &\quad + \|u\|_{W^{1,q}(\partial\Omega)}^{q-1} + 2 \max \left\{ 1, \int_{\partial\Omega} \tilde{\Lambda}_2(x, \alpha_2(x, u)) d\sigma \right\}. \end{aligned}$$

This shows $\mathcal{A}(U, \cdot) \in \mathcal{V}'$, for every $U \in \mathcal{V}$.

Next, let $U, V \in \mathcal{V}$. Then, using (2.11) and the fact that $\alpha_j(x, \cdot)$ are monotone nondecreasing, that is, $(\alpha_j(x, t) - \alpha_j(x, s))(t - s) \geq 0$, for all $t, s \in \mathbb{R}$, we obtain

$$\begin{aligned} &\mathcal{A}(U, U - V) - \mathcal{A}(V, U - V) \\ &= \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla (u - v) dx + \int_{\Omega} (|u|^{p-2} u - |v|^{p-2} v) (u - v) dx \\ &\quad + \int_{\Omega} (\alpha_1(x, u) - \alpha_1(x, v)) (u - v) dx + \int_{\partial\Omega} (|u|^{q-2} u - |v|^{q-2} v) (u - v) d\sigma \\ &\quad + \int_{\partial\Omega} (|\nabla_{\Gamma} u|^{q-2} \nabla_{\Gamma} u - |\nabla_{\Gamma} v|^{q-2} \nabla_{\Gamma} v) \cdot \nabla_{\Gamma} (u - v) d\sigma \\ &\quad + \int_{\partial\Omega} (\alpha_2(x, u) - \alpha_2(x, v)) (u - v) d\sigma \\ &\geq \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u - v)|^2 dx + \int_{\Omega} (|u| + |v|)^{p-2} |u - v|^2 dx \\ &\quad + \int_{\partial\Omega} (|\nabla_{\Gamma} u| + |\nabla_{\Gamma} v|)^{p-2} |\nabla_{\Gamma} (u - v)|^2 d\sigma + \int_{\partial\Omega} (|u| + |v|)^{p-2} |u - v|^2 d\sigma \\ &\geq 0. \end{aligned} \quad (5.7)$$

This shows that \mathcal{A} is monotone. The estimate (5.7) also shows that

$$\mathcal{A}(U, U - V) - \mathcal{A}(V, U - V) > 0,$$

for all $U, V \in \mathcal{V}$ with $U \neq V$, that is, $u \neq v$ or $u|_{\partial\Omega} \neq v|_{\partial\Omega}$. Thus, \mathcal{A} is strictly monotone.

The continuity of the norm function and the continuity of $\alpha_j(x, \cdot)$, $j = 1, 2$ imply that \mathcal{A} is hemicontinuous.

Finally, since Λ_j and $\tilde{\Lambda}_j$ satisfy the (Δ_2^0) -condition, from Proposition 2.10 and Corollary 2.11, it follows

$$\lim_{\|u\|_{\Lambda_1, \Omega} \rightarrow +\infty} \frac{\int_{\Omega} u \alpha_1(x, u) dx}{\|u\|_{\Lambda_1, \Omega}} = +\infty, \text{ and } \lim_{\|u\|_{\Lambda_2, \partial\Omega} \rightarrow +\infty} \frac{\int_{\partial\Omega} u \alpha_2(x, u) d\sigma}{\|u\|_{\Lambda_2, \partial\Omega}} = +\infty.$$

Consequently, we deduce

$$\lim_{\|U\|_{\mathcal{V}} \rightarrow +\infty} \frac{\mathcal{A}(U, U)}{\|U\|_{\mathcal{V}}} = +\infty, \quad (5.8)$$

which shows that \mathcal{A} is coercive. The proof of the lemma is finished. \square

The following result is concerned with the existence and uniqueness of weak solutions to problem (5.1).

Theorem 5.5. *Assume Assumptions 5.1 and 5.2. Let $1 < p, q < \infty$, $p_1 \geq p^*$ and $q_1 \geq q^*$, where $p^* := p/(p-1)$ and $q^* := q/(q-1)$. Then for every $(f, g) \in X^{p_1, q_1}(\overline{\Omega}, \mu)$, there exists a unique function $U \in \mathcal{V}$ which is a weak solution to (5.1).*

Proof. Let $\langle \cdot, \cdot \rangle$ denote the duality between \mathcal{V} and \mathcal{V}' . Then, from Lemma 5.4, it follows that for each $U \in \mathcal{V}$, there exists $A(U) \in \mathcal{V}'$ such that

$$\mathcal{A}(U, V) = \langle A(U), V \rangle,$$

for every $V \in \mathcal{V}$. Hence, this relation defines an operator $A : \mathcal{V} \rightarrow \mathcal{V}'$, which is bounded by (5.6). Exploiting Lemma 5.4 once again, it is easy to see that A is monotone and coercive. It follows from Brodwer's theorem (see, e.g., [11, Theorem 5.3.22]), that $A(\mathcal{V}) = \mathcal{V}'$. Therefore, for every $F \in \mathcal{V}'$ there exists $U \in \mathcal{V}$ such that $A(U) = F$, that is, for every $V \in \mathcal{V}$,

$$\langle A(U), V \rangle = \mathcal{A}(U, V) = \langle V, F \rangle.$$

Since $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,q}(\partial\Omega) \hookrightarrow L^q(\partial\Omega)$ with dense injection, by duality, we have $X^{p^*, q^*}(\overline{\Omega}, \mu) \hookrightarrow \mathcal{V}'$. Since Ω is bounded and $\sigma(\partial\Omega) < \infty$, we obtain that

$$X^{p_1, q_1}(\overline{\Omega}, \mu) \hookrightarrow X^{p^*, q^*}(\overline{\Omega}, \mu) \hookrightarrow \mathcal{V}'.$$

This shows the existence of weak solutions. The uniqueness follows from the fact that \mathcal{A} is strictly monotone (cf. Lemma 5.4). This completes the proof of the theorem. \square

Corollary 5.6. *Let the assumptions of Theorem 5.5 be satisfied. Let*

$$p_h := \frac{Np}{N(p-1)+p}, \quad q_h := \frac{p(N-1)}{N(p-1)}, \quad \text{and} \quad q_k := \frac{q(N-1)}{N(q-1)+1}. \quad (5.9)$$

- (a) *Let $1 < p < N$, $1 < q < p(N-1)/N$, $p_1 \geq p_h$ and $q_1 \geq q_k$. Then for every $(f, g) \in X^{p_1, q_1}(\Omega, \mu)$, there exists a function $U \in \mathcal{V}$ which is the unique weak solution to (5.1).*
- (b) *Let $1 < q < N-1$, $1 < p < Nq/(N-1)$, $p_1 \geq p_h$ and $q_1 \geq q_h$. Then for every $(f, g) \in X^{p_1, q_1}(\Omega, \mu)$, there exists a function $U \in \mathcal{V}$ which is the unique weak solution to (5.1).*

Proof. We first prove (1). Let $1 < p < N$ and $1 < q < p(N-1)/N$ and let $p_1 \geq p_h$ and $q_1 \geq q_k$, where p_h and q_k are given by (5.9). Let $p_s := Np/(N-p)$ and $q_t := (N-1)q/(N-1-q)$. Since $W^{1,p}(\Omega) \hookrightarrow L^{p_s}(\Omega)$ and $W^{1,q}(\partial\Omega) \hookrightarrow L^{q_t}(\partial\Omega)$ with dense injection, then by duality, $X^{p_h, q_k}(\overline{\Omega}, \mu) \hookrightarrow \mathcal{V}'$, where $1/p_s + 1/p_h = 1$ and $1/q_t + 1/q_k = 1$. Since $\mu(\overline{\Omega}) < \infty$, we have that

$$X^{p_1, q_1}(\overline{\Omega}, \mu) \hookrightarrow X^{p_h, q_k}(\overline{\Omega}, \mu) \hookrightarrow \mathcal{V}'.$$

Hence, for every $F := (f, g) \in X^{p_1, q_1}(\overline{\Omega}, \mu) \hookrightarrow \mathcal{V}'$, there exists $U \in \mathcal{V}$ such that for every $V \in \mathcal{V}$,

$$\langle A(U), V \rangle = \mathcal{A}(U, V) = \int_{\Omega} f v \, dx + \int_{\partial\Omega} g v \, d\sigma.$$

The uniqueness of the weak solution follows again from the fact that \mathcal{A} is strictly monotone.

In order to prove the second part, we use the embeddings $W^{1,p}(\Omega) \hookrightarrow L^{p_s}(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^{q_s}(\partial\Omega)$ and proceed exactly as above. We omit the details. \square

5.3. Properties of the solution operator of the perturbed equation. In the sequel, we establish some interesting properties of the solution operator A to problem (5.1). We begin by assuming the following.

Assumption 5.7. Suppose that α_j , $j = 1, 2$, satisfy the following conditions:

$$\begin{cases} \text{there are constants } c_j \in (0, 1] \text{ such that} \\ c_j |\alpha_j(x, \xi - \eta)| \leq |\alpha_j(x, \xi) - \alpha_j(x, \eta)| \text{ for all } \xi, \eta \in \mathbb{R}. \end{cases} \quad (5.10)$$

Theorem 5.8. Assume Assumptions 5.1, 5.2 and 5.7. Let $p, q \geq 2$ and let $A : \mathcal{V} \rightarrow \mathcal{V}'$ be the continuous and bounded operator constructed in the proof of Theorem 5.5. Then A is injective and hence, invertible and its inverse A^{-1} is also continuous and bounded.

Proof. First, we remark that, since

$$(\alpha_j(x, t) - \alpha_j(x, s))(t - s) \geq 0, \text{ for all } t, s \in \mathbb{R},$$

for λ_N -a.e. $x \in \Omega$ if $j = 1$ and σ -a.e. $x \in \partial\Omega$ if $j = 2$, it follows from (5.10) that, for all $t, s \in \mathbb{R}$,

$$(\alpha_j(x, t) - \alpha_j(x, s))(t - s) \geq c_j \alpha_j(x, t - s) \cdot (t - s). \quad (5.11)$$

Let $U, V \in \mathcal{V}$ and $p, q \in [2, \infty)$. Then, exploiting (2.12), (5.11) and the (Δ_2) -condition, we obtain

$$\begin{aligned} \langle A(U) - A(V), U - V \rangle &= \mathcal{A}(U, U - V) - \mathcal{A}(V, U - V) \\ &= \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla (u - v) dx + \int_{\Omega} (|u|^{p-2} u - |v|^{p-2} v) (u - v) dx \\ &\quad + \int_{\Omega} (\alpha_1(x, u) - \alpha_1(x, v)) (u - v) dx + \int_{\partial\Omega} (|\nabla_{\Gamma} u|^{q-2} \nabla_{\Gamma} u - |\nabla_{\Gamma} v|^{q-2} \nabla_{\Gamma} v) \cdot \nabla_{\Gamma} (u - v) d\sigma \\ &\quad + \int_{\partial\Omega} (|u|^{q-2} u - |v|^{q-2} v) (u - v) d\sigma + \int_{\partial\Omega} (\alpha_2(x, u) - \alpha_2(x, v)) (u - v) d\sigma \\ &\geq \|u - v\|_{W^{1,p}(\Omega)}^p + c_1 \int_{\Omega} \Lambda_1(x, u - v) dx + \|u - v\|_{W^{1,q}(\partial\Omega)}^q + c_2 \int_{\partial\Omega} \Lambda_2(x, u - v) d\sigma. \end{aligned} \quad (5.12)$$

This implies that $\langle A(U) - A(V), U - V \rangle > 0$, for all $U, V \in \mathcal{V}$ with $U \neq V$ (that is, $u \neq v$, or $u|_{\partial\Omega} \neq v|_{\partial\Omega}$). Therefore, the operator A is injective and hence, A^{-1} exists. Since for every $U \in \mathcal{V}$,

$$\mathcal{A}(U, U) = \langle A(U), U \rangle \leq \|A(U)\|_{\mathcal{V}'} \|U\|_{\mathcal{V}},$$

from the coercivity of \mathcal{A} (see (5.8)), it is not difficult to see that

$$\lim_{\|U\|_{\mathcal{V}} \rightarrow +\infty} \|A(U)\|_{\mathcal{V}'} = +\infty. \quad (5.13)$$

Thus, $A^{-1} : \mathcal{V}' \rightarrow \mathcal{V}$ is bounded.

Next, we show that $A^{-1} : \mathcal{V}' \rightarrow \mathcal{V}$ is continuous. Assume that A^{-1} is not continuous. Then there is a sequence $F_n \in \mathcal{V}'$ with $F_n \rightarrow F$ in \mathcal{V}' and a constant $\delta > 0$ such that

$$\|A^{-1}(F_n) - A^{-1}(F)\|_{\mathcal{V}} \geq \delta, \quad (5.14)$$

for all $n \in \mathbb{N}$. Let $U_n := A^{-1}(F_n)$ and $U = A^{-1}(F)$. Since $\{F_n\}$ is a bounded sequence and A^{-1} is bounded, we have that $\{U_n\}$ is bounded in \mathcal{V} . Thus, we can select a subsequence, which we still denote by $\{U_n\}$, which converges weakly to some function $V \in \mathcal{V}$. Since $A(U_n) - A(V) \rightarrow F - A(V)$ strongly in \mathcal{V} and $U_n - V$ converges weakly to zero in \mathcal{V} , we deduce

$$\lim_{n \rightarrow \infty} \langle A(U_n) - A(V), U_n - V \rangle = 0. \quad (5.15)$$

From (5.12) and (5.15), it follows that

$$\lim_{n \rightarrow \infty} \|u_n - v\|_{W^{1,p}(\Omega)} = 0 \text{ and } \lim_{n \rightarrow \infty} \int_{\Omega} \Lambda_1(x, u_n - v) dx = 0,$$

while

$$\lim_{n \rightarrow \infty} \|u_n - v\|_{W^{1,q}(\partial\Omega)} = 0 \text{ and } \lim_{n \rightarrow \infty} \int_{\partial\Omega} \Lambda_2(x, u_n - v) d\sigma = 0.$$

Therefore, $U_n \rightarrow V$ strongly in \mathcal{V} . Since A is continuous and

$$F_n = A(U_n) \rightarrow A(V) = F = A(U)$$

it follows from the injectivity of A , that $U = V$. This shows that

$$\lim_{n \rightarrow \infty} \|A^{-1}(F_n) - A^{-1}(F)\|_{\mathcal{V}} = \lim_{n \rightarrow \infty} \|U_n - U\|_{\mathcal{V}} = 0,$$

which contradicts (5.14). Hence, $A^{-1} : \mathcal{V}' \rightarrow \mathcal{V}$ is continuous. The proof is finished. \square

Corollary 5.9. *Let the assumptions of Theorem 5.8 be satisfied. Let p_h, q_h and q_k be as in (5.9) and let $A : \mathcal{V} \rightarrow \mathcal{V}'$ be the continuous and bounded operator constructed in the proof of Theorem 5.5.*

- (a) *If $2 \leq p < N$, $2 \leq q < p(N-1)/N$, $p_1 \geq p_h$ and $q_1 \geq q_k$, then $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow X^{p_s, q_t}(\overline{\Omega}, \mu)$ is continuous and bounded. Moreover, $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow \mathcal{V} \cap X^{r, s}(\overline{\Omega}, \mu)$ is compact for every $r \in (1, p_s)$ and $s \in (1, q_s)$.*
- (b) *If $2 \leq q < N-1$, $2 \leq p < qN/(N-1)$, $p_1 \geq p_h$ and $q_1 \geq q_h$, then the operator $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow X^{p_s, q_s}(\overline{\Omega}, \mu)$ is continuous and bounded. Moreover, $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow \mathcal{V} \cap X^{r, s}(\overline{\Omega}, \mu)$ is compact for every $r \in (1, p_s)$ and $s \in (1, q_s)$.*

Proof. We only prove the first part. The second part of the proof follows by analogy and is left to the reader. Let $2 \leq p < N$, $2 \leq q < p(N-1)/N$, $p_1 \geq p_h$ and $q_1 \geq q_k$ and let $F \in X^{p_1, q_1}(\overline{\Omega}, \mu)$. Proceeding exactly as in the proof of Theorem 5.8, we obtain

$$\|A^{-1}(F)\|_{p_s, q_t} \leq C_1 \|A^{-1}(F)\|_{\mathcal{V}} \leq C \|F\|_{\mathcal{V}'} \leq C_2 \|F\|_{p_1, q_1}.$$

Hence, the operator $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow X^{p_s, q_t}(\overline{\Omega}, \mu)$ is bounded. Finally, using the facts that $X^{p_1, q_1}(\overline{\Omega}, \mu) \hookrightarrow \mathcal{V}'$, $A^{-1} : \mathcal{V}' \rightarrow \mathcal{V}$ is continuous and $\mathcal{V} \hookrightarrow X^{p_s, q_t}(\overline{\Omega}, \mu)$, we easily deduce that $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow X^{p_s, q_t}(\overline{\Omega}, \mu)$ is continuous.

Now, let $1 < r < p_s$ and $1 < s < q_s$. Since the injection $\mathcal{V} \hookrightarrow X^{r, s}(\overline{\Omega}, \mu)$ is compact, then by duality, the injection $X^{r', s'}(\overline{\Omega}, \mu) \hookrightarrow (\mathcal{V})^*$ is compact for every $r' > p'_s = p_h$ and $s' > q'_s = q_h$. This, together with the fact that $A^{-1} : (\mathcal{V})^* \rightarrow \mathcal{V}$ is continuous and bounded, imply that $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow \mathcal{V}$ is compact for every $p_1 > p_h$ and $q_1 > q_h$.

It remains to show that A^{-1} is also compact as a map into $X^{r, s}(\overline{\Omega}, \mu)$ for every $r \in (1, p_s)$ and $s \in (1, q_s)$. Since A^{-1} is bounded, we have to show that the image of every bounded set $\mathcal{B} \subset X^{p_1, q_1}(\overline{\Omega}, \mu)$ is relatively compact in $X^{r, s}(\overline{\Omega}, \mu)$ for every $r \in (1, p_s)$ and $s \in (1, q_s)$. Let U_n be a sequence in $A^{-1}(\mathcal{B})$. Let $F_n = A(U_n) \in \mathcal{B}$. Since \mathcal{B} is bounded, then the sequence F_n is bounded. Since A^{-1} is compact as a map into \mathcal{V} , it follows that there is a subsequence F_{n_k} such that $A^{-1}(F_{n_k}) \rightarrow U \in \mathcal{V}$. We may assume that $U_n = A^{-1}(F_n) \rightarrow U$ in \mathcal{V} and hence, in $X^{p, p}(\overline{\Omega}, \mu)$. It remains to show that $U_n \rightarrow U$ in $X^{r, s}(\overline{\Omega}, \mu)$. Let $r \in [p, p_s)$ and $s \in [p, q_s)$. Since $U_n := (u_n, u_n|_{\partial\Omega})$ is bounded in $X^{p_s, q_s}(\overline{\Omega}, \mu)$, a standard interpolation inequality shows that there exists $\tau \in (0, 1)$ such that

$$\|U_n - U_m\|_{r, s} \leq \|U_n - U_m\|_{p, p}^{\tau} \|U_n - U_m\|_{p_s, q_s}^{1-\tau} \leq C \|U_n - U_m\|_{p, p}^{\tau}.$$

As U_n converges in $X^{p, p}(\overline{\Omega}, \mu)$, it follows from the preceding inequality that U_n is a Cauchy sequence in $X^{r, s}(\overline{\Omega}, \mu)$ and therefore converges in $X^{r, s}(\overline{\Omega}, \mu)$. Hence, $A^{-1} : X^{p_1, q_1}(\overline{\Omega}, \mu) \rightarrow$

$\mathcal{V} \cap X^{r,s}(\overline{\Omega}, \mu)$ is compact for every $r \in [p, p_s)$ and $s \in [p, q_s)$. The case $r, s \in (1, p)$ follows from the fact that $X^{p,p}(\overline{\Omega}, \mu) \hookrightarrow X^{r,s}(\overline{\Omega}, \mu)$ and the proof is finished \square

5.4. Statement and proof of the main result. We will now establish under what conditions the operator A^{-1} maps $X^{p_1, q_1}(\overline{\Omega}, \mu)$ boundedly and continuously into $X^\infty(\overline{\Omega}, \mu)$. The following is the main result of this section.

Theorem 5.10. *Let the assumptions of Theorem 5.8 be satisfied.*

(a) *Suppose $2 \leq p < N$ and $2 \leq q < \infty$. Let*

$$p_1 > \frac{p_s}{p_s - p} = \frac{N}{p} \text{ and } q_1 > \frac{q_s}{q_s - p} = \frac{N-1}{p-1}.$$

Let $f \in L^{p_1}(\Omega)$, $g \in L^{q_1}(\partial\Omega)$ and $U, V \in \mathcal{V}$ be such that for every function $\Phi = (\varphi, \varphi|_{\partial\Omega}) \in \mathcal{V}$,

$$\mathcal{A}(U, \Phi) - \mathcal{A}(V, \Phi) = \int_{\Omega} f \varphi \, dx + \int_{\partial\Omega} g \varphi \, d\sigma. \quad (5.16)$$

Then there is a constant $C = C(N, p, q, \Omega) > 0$ such that

$$\|U - V\|_{\infty}^{p-1} \leq C(\|f\|_{p_1, \Omega} + \|g\|_{q_1, \partial\Omega}).$$

(b) *Suppose $2 \leq p = q < N-1$. Let*

$$p_1 > \frac{p_s}{p_s - p} = \frac{N}{p} \text{ and } q_1 > \frac{p_t}{p_t - p} = \frac{N-1}{p}.$$

Let $f \in L^{p_1}(\Omega)$, $g \in L^{q_1}(\partial\Omega)$ and $U, V \in \mathcal{V}$ satisfy (5.16). Then there is a constant $C = C(N, p, q, \Omega) > 0$ such that

$$\|U - V\|_{\infty}^{p-1} \leq C(\|f\|_{p_1, \Omega} + \|g\|_{q_1, \partial\Omega}).$$

Proof. Let $U, V \in \mathcal{V}$ satisfy (5.16). Let $k \geq 0$ be a real number and set

$$w_k := (|u - v| - k)^+ \operatorname{sgn}(u - v) \quad W_k := (w_k, w_k|_{\partial\Omega}) \quad \text{and} \quad w := |u - v|.$$

Let $A_k := \{x \in \overline{\Omega} : |w(x)| \geq k\}$, and $A_k^+ := \{x \in \overline{\Omega} : w(x) \geq k\}$, $A_k^- := \{x \in \overline{\Omega} : w(x) \leq -k\}$. Clearly $W_k \in \mathcal{V}$ and $A_k = A_k^+ \cup A_k^-$. We claim that there exists a constant $C > 0$ such that

$$C\mathcal{A}(W_k, W_k) \leq \mathcal{A}(U, W_k) - \mathcal{A}(V, W_k), \quad (5.17)$$

for all $U, V \in \mathcal{V}$. Using the definition of the form \mathcal{A} , we have

$$\begin{aligned} & \mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) \\ &= \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla w_k \, dx + \int_{\Omega} (|u|^{p-2} u - |v|^{p-2} v) w_k \, dx \\ &+ \int_{\Omega} (\alpha_1(x, u) - \alpha_2(x, v)) w_k \, dx + \int_{\partial\Omega} (|u|^{q-2} u - |v|^{q-2} v) w_k \, d\sigma \\ &+ \int_{\partial\Omega} (|\nabla_{\Gamma} u|^{p-2} \nabla_{\Gamma} u - |\nabla_{\Gamma} v|^{p-2} \nabla_{\Gamma} v) \cdot \nabla_{\Gamma} w_k \, d\sigma + \int_{\partial\Omega} (\alpha_2(x, u) - \alpha_2(x, v)) w_k \, d\sigma. \end{aligned} \quad (5.18)$$

Since $\nabla w_k = \begin{cases} \nabla(u-v) & \text{in } A(k), \\ 0 & \text{otherwise,} \end{cases}$ we can rewrite (5.18) as follows:

$$\begin{aligned} \mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) &= \int_{A(k) \cap \Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla(u-v) dx \\ &+ \int_{A(k) \cap \partial \Omega} (|\nabla_\Gamma u|^{q-2} \nabla_\Gamma u - |\nabla_\Gamma v|^{q-2} \nabla_\Gamma v) \cdot \nabla_\Gamma(u-v) d\sigma \\ &+ \lambda \int_{A(k) \cap \Omega} (|u|^{p-2} u - |v|^{p-2} v) w_k dx + \int_{A(k) \cap \Omega} (\alpha_1(x, u) - \alpha_1(x, v)) w_k dx \\ &+ \int_{A(k) \cap \partial \Omega} (\alpha_2(x, u) - \alpha_2(x, v)) w_k d\sigma. \end{aligned} \quad (5.19)$$

Exploiting inequality (2.12), from (5.19) and (5.11), we deduce

$$\begin{aligned} \mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) &\geq \int_{A(k) \cap \Omega} (|\nabla w_k|^p + |w_k|^p) dx + \int_{A(k) \cap \Omega} c_1 \alpha_1(x, w_k) w_k dx \\ &+ \int_{A(k) \cap \Omega} (|u|^{p-2} u w_k - |v|^{p-2} v w_k - |w_k|^p) dx \\ &+ \int_{A(k) \cap \Omega} (\alpha_1(x, u) - \alpha_1(x, v) - c_1 \alpha_1(x, w_k)) w_k dx \\ &+ \int_{A(k) \cap \partial \Omega} (|\nabla_\Gamma w_k|^q + |w_k|^q) d\sigma + \int_{A(k) \cap \partial \Omega} c_2 \alpha_2(x, w_k) w_k d\sigma \\ &+ \int_{A(k) \cap \partial \Omega} (|u|^{q-2} u w_k - |v|^{q-2} v w_k - |w_k|^q) d\sigma \\ &+ \int_{A(k) \cap \partial \Omega} (\alpha_2(x, u) - \alpha_2(x, v) - c_2 \alpha_2(x, w_k)) w_k d\sigma \\ &\geq C \mathcal{A}(W_k, W_k) + \int_{A(k) \cap \Omega} (|u|^{p-2} u w_k - |v|^{p-2} v w_k - |w_k|^p) dx \\ &+ \int_{A(k) \cap \Omega} (\alpha_1(x, u) - \alpha_1(x, v) - c_1 \alpha_1(x, w_k)) w_k dx \\ &+ \int_{A(k) \cap \partial \Omega} (|u|^{q-2} u w_k - |v|^{q-2} v w_k - |w_k|^q) d\sigma \\ &+ \int_{A(k) \cap \partial \Omega} (\alpha_2(x, u) - \alpha_2(x, v) - c_2 \alpha_2(x, w_k)) w_k d\sigma, \end{aligned} \quad (5.20)$$

where c_1, c_2 are the constants from (5.11). Using (5.10) and the fact that $\alpha_j(x, \cdot)$ are strictly increasing, for $x \in A_k^+$, we have

$$\begin{aligned} c_j \alpha_j(x, w_k(x)) &= c_j \alpha_j(x, u(x) - v(x) - k) \leq c_j \alpha_j(x, u(x) - v(x)) \\ &\leq \alpha_j(x, u(x)) - \alpha_j(x, v(x)). \end{aligned}$$

Multiplying this inequality by $w_k(x) \geq 0$, $x \in A_k^+$, yields

$$(\alpha_j(x, u(x)) - \alpha_j(x, v(x)) - c_j \alpha_j(x, w_k(x))) w_k(x) \geq 0. \quad (5.21)$$

Similarly, for $x \in A_k^-$,

$$\begin{aligned} c_j \alpha_j(x, w_k(x)) &= c_j \alpha_j(x, u(x) - v(x) + k) \geq c_j \alpha_j(x, u(x) - v(x)) \\ &\geq \alpha_j(x, u(x)) - \alpha_j(x, v(x)). \end{aligned}$$

Hence, multiplying this inequality by $w_k(x) \leq 0$, we get

$$(\alpha_j(x, u(x)) - \alpha_j(x, v(x)) - c_j \alpha_j(x, w_k(x))) w_k(x) \geq 0, \quad (5.22)$$

for all $x \in A_k^-$. Hence, on account of (5.21) and (5.22), from (5.20) we obtain the required estimate of (5.17).

(a) To prove this part, note that from Definition 5.3 it is clear that,

$$\|w_k\|_{W^{1,p}(\Omega)}^p \leq \mathcal{A}(W_k, W_k). \quad (5.23)$$

Let $f \in L^{p_1}(\Omega)$ and $g \in L^{q_1}(\partial\Omega)$ with

$$p_1 > \frac{p_s}{p_s - p} = \frac{N}{p} \quad \text{and} \quad q_1 > \frac{q_s}{q_s - p} = \frac{N-1}{p-1},$$

and let $B \subset \overline{\Omega}$ be any μ -measurable set. We claim that there exists a constant $C \geq 0$ such that, for every $F \in X^{p_1, q_1}(\overline{\Omega}, \mu)$ and $\varphi \in W^{1,p}(\Omega)$, we have

$$\|F\varphi 1_B\|_{1,1} \leq C \|F\|_{p_1, q_1} \|\varphi\|_{W^{1,p}(\Omega)} \|\chi_B\|_{p_3, q_3}, \quad (5.24)$$

where p_3 and q_3 are such that $1/p_3 + 1/p_1 + 1/p_s = 1$ and $1/q_3 + 1/q_1 + 1/q_s = 1$. In fact, note that if $n \in \mathbb{N}$ and $p_i, q_i \in [1, \infty]$, $(i = 1, \dots, n)$ are such that

$$\sum_{i=1}^n \frac{1}{p_i} = \sum_{i=1}^n \frac{1}{q_i} = 1,$$

and, if $F_i \in X^{p_i, q_i}(\overline{\Omega}, \mu)$, $(i = 1, \dots, n)$, then by Hölder's inequality,

$$\|\prod_{i=1}^n F_i\|_{1,1} \leq \prod_{i=1}^n \|F_i\|_{p_i, q_i}. \quad (5.25)$$

Since $W^{1,p}(\Omega) \hookrightarrow X^{p_s, q_s}(\overline{\Omega}, \mu)$, (5.24) follows immediately from (5.25) and the claim (5.24) is proved. Next, it follows from (5.24), that

$$\begin{aligned} \int_{\Omega} F W_k d\mu &= \|F W_k\|_{1,1} = \|F W_k \chi_{A_k}\|_{1,1} \\ &\leq \|F\|_{p_1, q_1} \|w_k\|_{W^{1,p}(\Omega)} \|\chi_{A_k}\|_{p_3, q_3}, \end{aligned}$$

where we recall that $1/p_3 = (1 - 1/p_s - 1/p_1) > (p-1)/p_s$ and $q_3 < q_s/(p-1)$. Therefore, for every $k \geq 0$,

$$\mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) \leq \|F\|_{p_1, q_1} \|w_k\|_{W^{1,p}(\Omega)} \|\chi_{A_k}\|_{p_3, q_3},$$

which together with estimate (5.17) yields the desired inequality

$$C \mathcal{A}(W_k, W_k) \leq \mathcal{A}(U, W_k) - \mathcal{B}(V, W_k) \leq \|F\|_{p_1, q_1} \|w_k\|_{W^{1,p}(\Omega)} \|\chi_{A_k}\|_{p_3, q_3}, \quad (5.26)$$

It follows from (5.23) and (5.26), that for every $k > 0$,

$$\begin{aligned} C \|w_k\|_{W^{1,p}(\Omega)}^p &\leq \mathcal{A}(W_k, W_k) \leq \mathcal{A}(U, W_k) - \mathcal{A}(V, W_k) \\ &\leq \|F\|_{p_1, q_1} \|w_k\|_{W^{1,p}(\Omega)} \|\chi_{A_k}\|_{p_3, q_3}. \end{aligned}$$

Hence, for every $k > 0$, $\|w_k\|_{W^{1,p}(\Omega)}^{p-1} \leq C_1 \|\chi_{A_k}\|_{p_3, q_3}$. Using the fact $W^{1,p}(\Omega) \hookrightarrow X^{p_s, q_s}(\overline{\Omega}, \mu)$, we obtain for every $k > 0$, that

$$\|w_k\|_{p_s, q_s}^{p-1} \leq C \|F\|_{p_1, q_1} \|\chi_{A_k}\|_{p_3, q_3}.$$

Let $h > k$. Then $A_h \subset A_k$ and on A_h the inequality $|w_k| \geq (h-k)$ holds. Therefore,

$$\|(h-k)\chi_{A_h}\|_{p_s, q_s}^{p-1} \leq C \|F\|_{p_1, q_1} \|\chi_{A_k}\|_{p_3, q_3},$$

which shows that

$$\|\chi_{A_h}\|_{p_s, q_s}^{p-1} \leq C \|F\|_{p_1, q_1} (h-k)^{-(p-1)} \|\chi_{A_k}\|_{p_3, q_3}. \quad (5.27)$$

Let $C_3 := \|1_{\overline{\Omega}}\|_{p_s, q_s}$, and

$$\delta := \min \left\{ \frac{p_s}{p_3}, \frac{q_s}{p_3} \right\} > p-1, \quad \delta_0 := \frac{\delta}{p-1} > 1.$$

Then

$$\begin{aligned} \|C_3^{-p_s/p_3} \chi_{A_k}\|_{\Omega, p_3} &= \|C_3^{-1} \chi_{A_k}\|_{\Omega, p_3}^{p_s/p_3} \leq \|C_3^{-1} \chi_{A_k}\|_{\Omega, p_s}^{\delta} \\ &\leq \|\chi_{A_k}\|_{p_s, q_s}^{\delta} C_3^{-\delta} \end{aligned} \quad (5.28)$$

and

$$\begin{aligned} \|C_3^{-q_s/q_3} \chi_{A_k}\|_{\partial\Omega, q_3} &= \|C_3^{-1} \chi_{A_k}\|_{\partial\Omega, q_3}^{q_s/q_3} \leq \|C_3^{-1} \chi_{A_k}\|_{\partial\Omega, q_s}^{\delta} \\ &\leq \|\chi_{A_k}\|_{p_s, q_s}^{\delta} C_3^{-\delta}. \end{aligned} \quad (5.29)$$

Choosing $C_{\Omega} := C_3^{p_s/p_3-\delta} + C_3^{q_s/q_3-\delta}$, from (5.28)-(5.29) we have

$$\|\chi_{A_k}\|_{p_3, q_3} \leq C_{\Omega} \|\chi_{A_k}\|_{p_s, q_s}^{\delta}. \quad (5.30)$$

Therefore, combining (5.27) with (5.30), we get

$$\begin{aligned} \|\chi_{A_h}\|_{p_s, q_s}^{p-1} &\leq C \|F\|_{p_1, q_1} (h-k)^{-(p-1)} \|\chi_{A_k}\|_{p_s, q_s}^{\delta} \\ &= C \|F\|_{p_1, q_1} (h-k)^{-(p-1)} \left[\|\chi_{A_k}\|_{p_s, q_s}^{p-1} \right]^{\delta_0}. \end{aligned} \quad (5.31)$$

Setting $\psi(h) := \|\chi_{A_h}\|_{p_s, q_s}^{p-1}$ in Lemma 2.13, on account of (5.31), we can find a constant C_2 (independent of F) such that

$$\|\chi_{A_K}\|_{p_s, q_s}^{p-1} = 0 \quad \text{with } K := C_2 \|F\|_{p_1, q_1}^{1/(p-1)}.$$

This shows that $\mu(A_K) = 0$, where $A_K = \{x \in \overline{\Omega} : |(u-v)(x)| \geq K\}$. Hence, we have $|u-v| \leq K$, μ -a.e. on $\overline{\Omega}$ so that

$$\|U - V\|_{\infty}^{p-1} \leq C_2 \|F\|_{p_1, q_1} = C_2 (\|f\|_{p_1, \Omega} + \|g\|_{q_1, \partial\Omega}),$$

which completes the proof of part (a).

(b) To prove this part, instead of (5.23) and (5.24), one uses $\|W_k\|_{\mathcal{V}_1}^p \leq \mathcal{A}(W_k, W_k)$ and $\|F\varphi 1_B\|_{1,1} \leq C \|F\|_{p_1, q_1} \|\varphi\|_{W^{1,p}(\Omega)} \|\chi_B\|_{p_3, q_3}$, (where p_3 and q_3 are such that $1/p_3 + 1/p_1 + 1/p_s = 1$ and $1/q_3 + 1/q_1 + 1/p_t = 1$) and the embedding $\mathcal{V} \hookrightarrow \mathcal{V}_1 \hookrightarrow X^{p_s, p_t}(\overline{\Omega}, \mu)$. The remainder of the proof follows as in the proof of part (a). \square

We conclude this section with the following example.

Example 5.11. Let $p \in [2, \infty)$, $b : \partial\Omega \rightarrow (0, \infty)$ be a strictly positive and σ -measurable function and let

$$\beta(x, \xi) := b(x) |\xi|^{p-2} \xi, \quad \xi \in \mathbb{R}.$$

Then, it is easy to verify that β satisfies Assumptions 5.1, 5.2 and 5.7 (see, e.g., [5, Example 4.17]).

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